8. Some Applications.

As discussed in the introduction, our result immediately implies that any connected triangulated closed polyhedral surface is generically rigid. Whether or not the surface can be embedded in $\mathbb{R}^3$ is irrelevant to our theorem. Thus we have the following corollary.

Corollary. *The* 1-*skeleton of any abstract triangulation of a* 2-*manifold is generically* 3-*rigid.*

Here are two applications of this corollary.

In Chapter 1 we introduced as quickly as possible the definition of generic rigidity, which is all that is necessary to prove the result. But there is another type of rigidity for frameworks that is related to the rigidity that we have discussed and which is important in its own right, namely static rigidity. Presented here are the barest details. For more information see [18], [17], and [10].

A load $L = (L_1, L_2, L_3, \ldots, L_v)$ on a 3-framework is an assignment of 3-vectors (forces) to the vertices. A resolution of the load L by the framework is an assignment of scalars ω_{ij} to the edges such that $\Sigma\omega_{ij}(p(v_i)-p(v_j)) + L_i = 0$ (sum over j with $\{v_i, v_j\}$ an edge) at each vertex v_i. An equilibrium load is load such that (i) $\Sigma L_i = 0$ (sum over all vertices), and (ii) $\Sigma L_i \times p(v_i) = 0$ (sum over all vertices). The 3-framework is statically rigid if every equilibrium load has a resolution.

The relationships between the types of rigidity we have mentioned are well known. We note three facts here. If a 3-framework is statically rigid, then it is rigid. A graph is generically 3-rigid if and only if it has some realization in 3-space which is statically rigid. Statically rigid realizations in 3-space of a given graph form an open set in $\mathbb{R}^{3v}$.

Our interest is in a theorem about the rigidity of a polar of a polyhedral surface. A polarity is a correspondence which relates each point to a plane (called the polar of the point) and each plane to a point (called the polar of the plane), such that the polar of the polar of a point (or plane) is itself and such that incidence is preserved (see for instance [5] and [9]). If we think of a polyhedral surface P as being a collection of planes and vertices with an incidence relation, then its polar P′ (under a given polarity) is the collection of vertices polar to the planes of P and planes polar to the vertices of P, along with the same incidence relation. To include collections that may not arise from surfaces, we take the following definitions, slightly modified, from Whiteley [20].

An abstract polyhedral surface is pair of sets V (vertices) and F (faces) with an incidence relation between them such that: (i) between any face and any vertex there is an alternating sequence of faces and vertices such that a face and a vertex adjacent in the sequence are incident; (ii) the faces f_i in F incident with a vertex v_j in V form a cycle of distinct faces $f_j^1, f_j^2, f_j^3, \ldots, f_j^s$ $(s \geq 3)$; (iii) the vertices v_j in V incident with a face i in F form a cycle of distinct vertices $v_1^i, v_2^i, v_3^i, \ldots, v_t^i$ $(t \geq 3)$. Each pair of vertices v_j, v_k

adjacent in a cycle for f^h are adjacent in exactly one other face f^i, and this pair of faces is adjacent in the two cycles of these vertices. Each pair of faces f^h, f^i adjacent in a cycle for v_j are adjacent in exactly one other vertex v_k, and this pair of vertices is adjacent in the two cycles of these faces. The two associated pairs $(f^h,f^i;v_j,v_k)$ or simply $(h,i;j,k)$ are called an edge of the abstract polyhedral surface.

An _image of an abstract polyhedral surface_ is a collection of points and planes in 3-space such that each point corresponds to a vertex and each plane to a face, and such that the point for a vertex lies on the plane for a face if the vertex and face are incident in the abstract polyhedral surface. On each of these planes is the polygonal disk bounded by the polygon formed by the cycle of vertices around the corresponding face. The union of these polygonal disks may be a surface, and thus a polyhedral surface; however, it might instead be self-intersecting.

A _sharp image of an abstract polyhedral surface_ is an image of an abstract polyhedral surface such that: (i) the faces at an edge are distinct planes; (ii) the vertices of an edge are distinct points; (iii) the vertices incident with each face span the plane; (iv) the faces incident with each vertex meet only in this point.

Note that a polar of an image of an abstract polyhedral surface is an image of an abstract polyhedral surface, and a polar of a sharp image of an abstract polyhedral surface is a sharp image of an abstract polyhedral surface.

A surface image framework is the underlying vertex-edge framework of a sharp image of an abstract polyhedral surface with additional edges across each face triangulating the face (no triangle collinear).

Whiteley [**20**] proved that a surface image framework on a sharp image of an abstract polyhedral surface is statically rigid if and only if one (and therefore all) surface image frameworks on a polar image of an abstract polyhedral surface is statically rigid. He noted that our result and the above theorem together imply the following theorem. We say an abstract polyhedral surface is simple if each of its vertices has degree 3.

Theorem. *For any simple abstract polyhedral surface there exists a sharp image on which any surface image framework is statically rigid, and therefore rigid.*

Proof. The dual of a simple abstract polyhedral surface X is a triangulated abstract polyhedral surface X′, the 1-skeleton G′ of which is generically 3-rigid. Hence there is a statically rigid realization F′ of G′ in 3-space, to which, since X′ is triangulated, naturally corresponds an image I′ of the abstract polyhedral surface X′. Note that F′ is a surface image framework on I′. Because the statically rigid realizations of G′ form an open set in $\mathbb{R}^{3v}$, we can choose the vertices of I′ so that I′ is sharp. Then any polar of I′ is a sharp image of X on which any surface image framework is statically rigid, and thus rigid. Q.E.D.

A completely different application of the corollary of our result has been pointed out by Kalai [**14**]. This corollary implies a couple of theorems about the lower bound on the number of simplices of a pseudomanifold (which Kalai remarked could also be proved by methods developed by Gromov [12]). He defines a pseudomanifold as follows.

A simplicial complex X is <u>pure</u> if all its maximal simplices have the same size. Maximal simplices of a pure simplicial complex are called <u>facets</u>. Two facets σ, τ of a pure simplicial complex are <u>adjacent</u> if they intersect in a maximal proper simplex of each. A pure simplicial complex X is <u>strongly connected</u> if for every two facets σ and τ of X, there is a sequence of facets $\sigma = \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_m = \tau$, such that σ_i and σ_{i+1} are adjacent, $0 \leq i < m$.

A <u>d-pseudomanifold</u> is a strongly connected d-dimensional simplicial complex, such that every (d-1)-simplex is contained in exactly two facets. A <u>d-pseudomanifold with boundary</u> is a strongly connected d-dimensional simplicial complex, such that every (d-1)-simplex is contained in at most two facets. For a d-pseudomanifold with boundary X, the <u>boundary</u> of X, ∂X, is the (d-1)-dimensional pure simplicial complex whose facets are those (d-1)-simplices of X which are included in a unique facet of X.

Let $f_k(X)$ be the number of k-dimensional simplices of an abstract simplicial complex X. Define $\varphi_k(n,d)$ by

$$\varphi_k(n,d) = \begin{cases} \binom{d}{k}n - \binom{d+1}{k+1}k & \text{for } 1 \leq k \leq d-2, \\ (d-1)n - (d+1)(d-2) & \text{for } k = d-1. \end{cases}$$

Kalai showed that the following theorem reduces to the above corollary.

<u>The Lower Bound Theorem for Pseudomanifolds</u>. *(i) If* X *is a* (d-1)-*pseudomanifold with* n *vertices, then* $f_k(X) \geq \varphi_k(n,d)$ *for* $1 \leq k \leq d-1$.

(ii) If equality holds for some k, $1 \le k \le d-1$, *then* X *is a stacked* $(d-1)$-*sphere.*

A <u>stacked</u> <u>d-ball</u> is defined recursively as follows. A simplex complex on a set of $d+1$ vertices is a stacked d-ball. Let X be a stacked d-ball and let σ be a $(d-1)$-simplex of X. Let v be some vertex not in X and let Y be the simplex complex on the set $\sigma * \{v\}$. Then $X \cup Y$ is also a stacked d-ball. A <u>stacked</u> <u>(d-1)-sphere</u> is the boundary of a stacked d-ball.

Define $\varphi_k^{\mathbf{b}}(n_i, n_b, d)$ by

$$\varphi_k^{\mathbf{b}}(n_i,n_b,d) = \begin{cases} \binom{d-1}{k} n_b + \binom{d}{k} n_i - \binom{d}{k+1} k & \text{for } 1 \le k \le d-2, \\ n_b + (d-1)n_i - (d-1) & \text{for } k = d-1. \end{cases}$$

Kalai remarked that the following theorem also follows from the above corollary.

<u>The Lower Bound Theorem for Pseudomanifolds with Boundary</u>. *Let* X *be a* $(d-1)$-*pseudomanifold with nonempty boundary. If* X *has* n_i *vertices in the interior and* n_b *vertices in the boundary then*

(i) $f_k(X) \ge \varphi_k^{\mathbf{b}}(n_i,n_b,d)$ *for* $1 \le k \le d-1$.

(ii) If equality holds for some k, $1 \le k \le d-1$, *then* X *is a stacked* $(d-1)$-*ball.*

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