1. Introduction.

This work resulted from trying to show that connected closed triangulated polyhedral surfaces are generically rigid. We begin by discussing that problem.

We approach it by considering the underlying edge-vertex frameworks of such surfaces. A d-framework is a simple graph G and a mapping p from the vertices of G into $\mathbb{R}^d$, and is denoted by (G,p); it may be called a realization of the graph G in $\mathbb{R}^d$. If (G,p) and (G,q) are two realizations of G in $\mathbb{R}^d$ we say they are equivalent if for every edge $\{v_i,v_j\}$ of G we have $|p(v_i)-p(v_j)| = |q(v_i)-q(v_j)|$, i.e. if corresponding edges of the frameworks have equal length. We say they are congruent if for every pair $\{v_i,v_j\}$ of vertices of G, whether it is an edge or not, we have $|p(v_i)-p(v_j)| = |q(v_i)-q(v_j)|$. For example, the 3-frameworks represented in Figure 1 are all equivalent but only A and B are congruent.

We say that the d-framework (G,p) is rigid if there exists some $\epsilon > 0$ such that, whenever (G,q) is another realization of G in $\mathbb{R}^d$ with $|p(v_i)-q(v_i)| < \epsilon$ for all vertices v_i of G and (G,q) is equivalent to (G,p), then (G,q) is congruent to (G,p). Otherwise we say (G,p) is flexible. Given a graph G with v vertices and a fixed dimension d, note that there is a correspondence between the elements of $\mathbb{R}^{dv}$ and the realizations of G in $\mathbb{R}^d$ by considering the sequence of coordinates of the image points of the vertices in any realization as an element of $\mathbb{R}^{dv}$. It

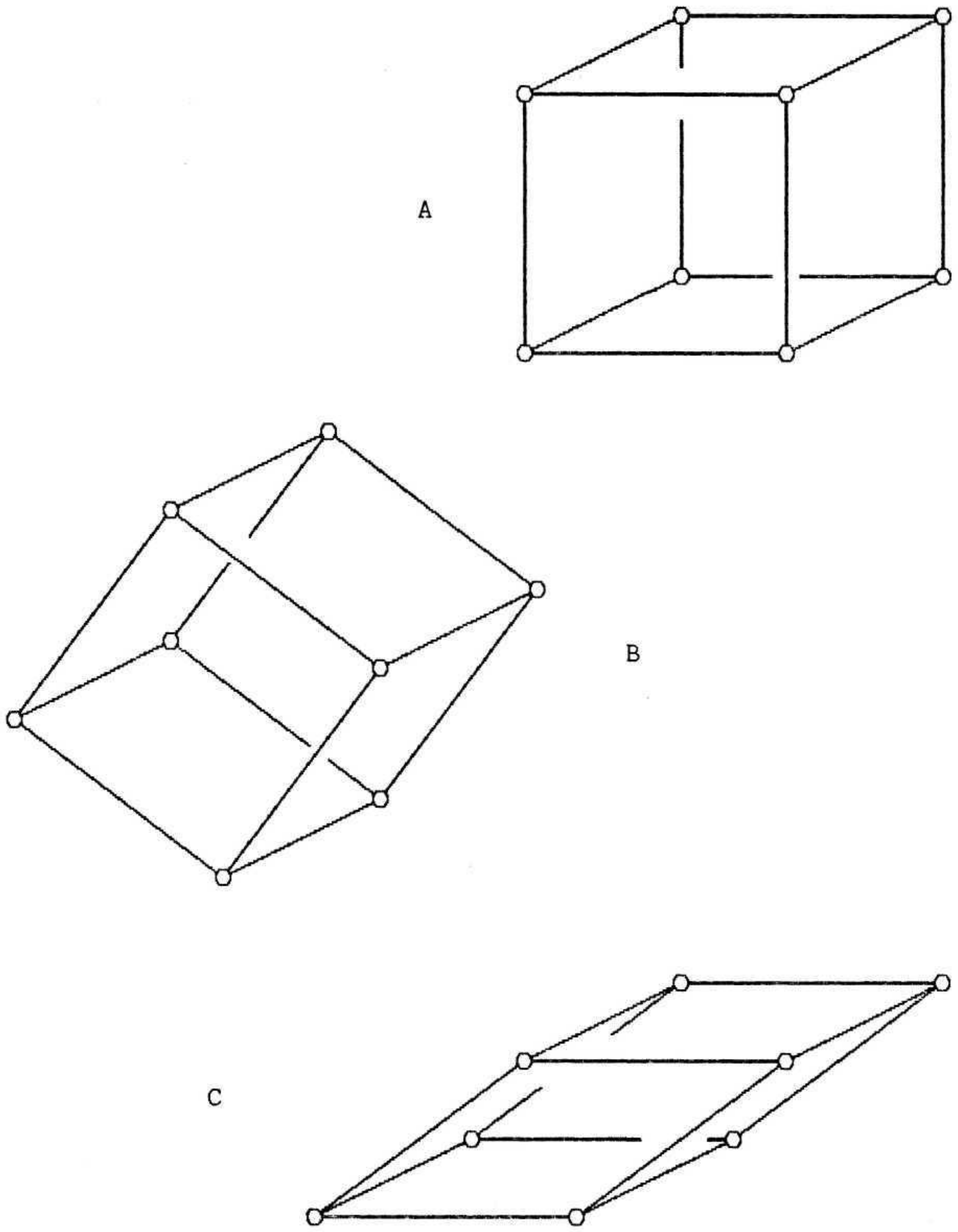


Figure 1. Equivalent frameworks.

has been shown that either the rigid realizations of G form an open dense set in $\mathbb{R}^{dv}$, or the flexible realizations of G form an open dense set in $\mathbb{R}^{dv}$ (see, for example, section 5 of Roth [17]). A graph G is said to be <u>generically</u> <u>d-rigid</u> if its rigid realizations form an open dense set in $\mathbb{R}^{dv}$. Otherwise we say G is <u>generically</u> <u>d-flexible</u>. We say that a d-framework is <u>generically</u> <u>rigid</u> if its graph is generically d-rigid, and we say it is <u>generically</u> <u>flexible</u> if its graph is generically d-flexible.

For example, the 3-frameworks represented in Figure 2 are realizations of a generically flexible graph, but the framework B is rigid. In either case the subframeworks on the vertices w, x, y, z and on v, w, x, z are rigid, but only when u is collinear with v and y is u held fixed.

As another example, the 3-frameworks represented in Figure 3 are generically rigid, but the framework B is flexible. As it flexes its vertices do not remain coplanar. Its flexibility was first studied by Bricard [4] and was used by Connelly to make a flexible closed polyhedral surface (see [7] and [8]).

Recall that a closed polyhedral surface is a surface in 3-space which is a finite union of polygons which pairwise intersect either exactly along a common edge, at a common vertex, or not at all. In evaluating the rigidity of such a structure, we think of it as a collection of rigid polygonal planar plates connected along their edges by hinges. In other words, we assume that vertices of a particular plate are held fixed relative to one another. Removing these plates leaves a framework which may not have edges between vertices formerly held a fixed distance apart. For example, if the faces of a cube are rigid plates,

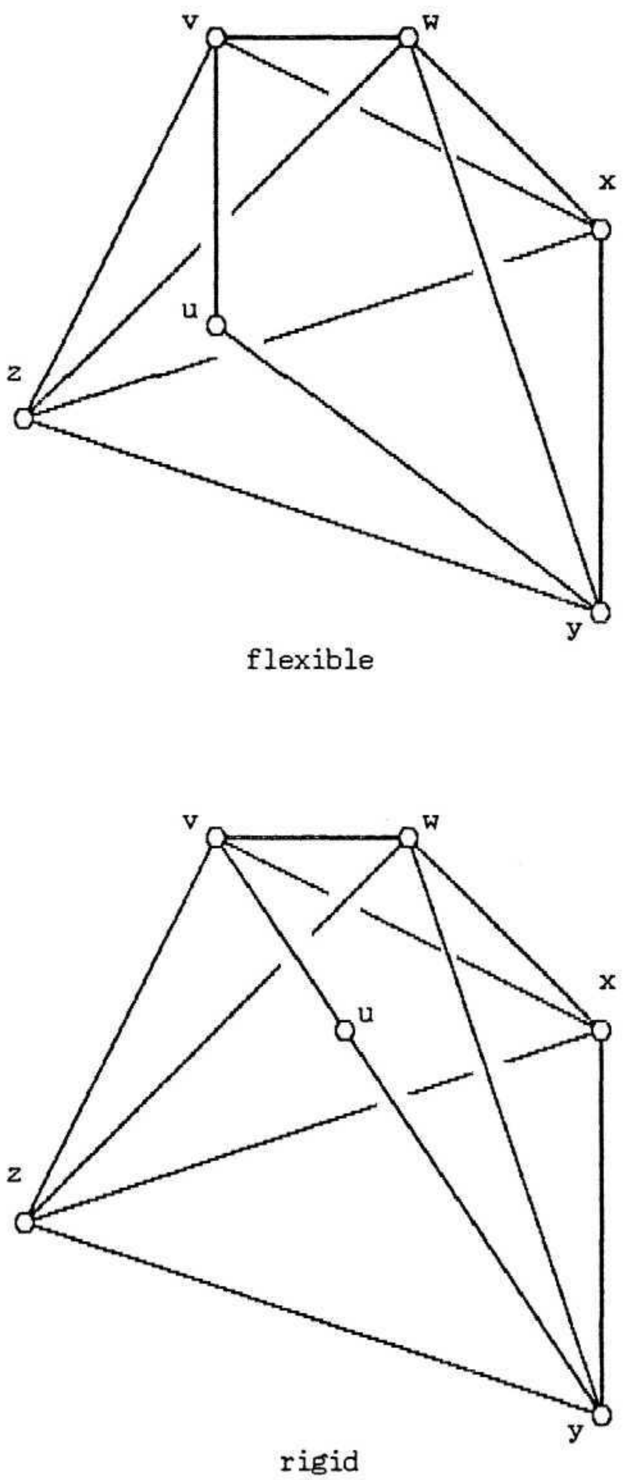


Figure 2. Realizations of a generically 3-flexible graph.

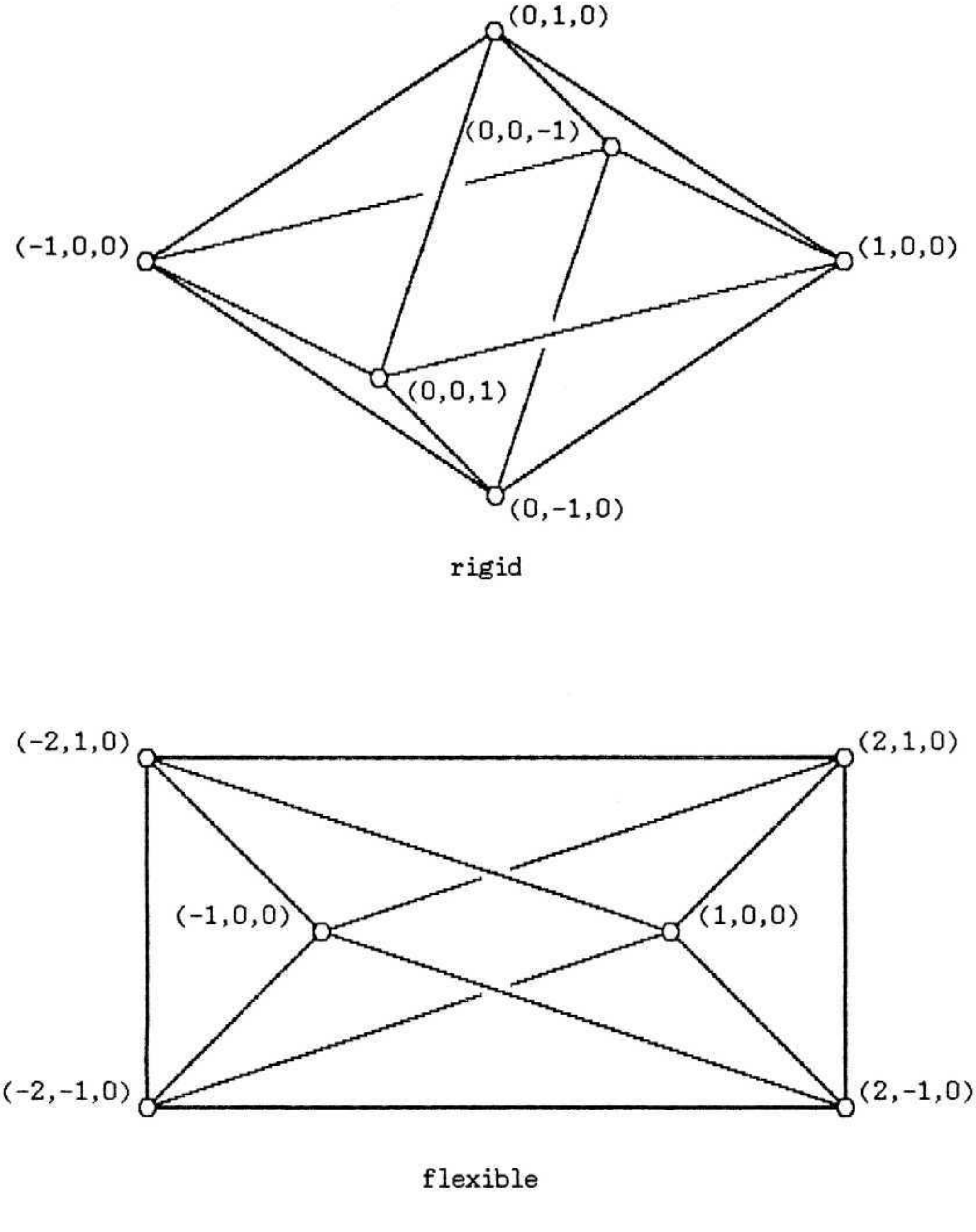


Figure 3. Realizations of a generically 3-rigid graph.

the cube is rigid; but if the plates are removed, the underlying framework is flexible. However, if all the plates of a surface are triangles then the rigidity is not affected by removing the plates. We therefore say that a closed triangulated polyhedral surface is rigid if its underlying edge-vertex framework is rigid; otherwise it is called flexible. We call it generically rigid if its framework is generically rigid and call it generically flexible otherwise.

Cauchy [6] showed that the surfaces of convex polyhedra are rigid in 1813, but it was not until 1974 that something could be said about the non-convex case. Gluck [10] showed that simply connected triangulated closed polyhedral surfaces are generically rigid; a few years later Connelly [7] found one that is flexible.

By applying Connelly's techniques, one can find flexible examples of triangulated closed polyhedral surfaces of any topological type; however, all known connected examples are generically rigid, and it was conjectured that all connected triangulated closed polyhedral surfaces are generically rigid. Whiteley [19] and Graver [11] independently showed that triangulated toroidal polyhedral surfaces are generically rigid by showing that the graphs of abstract triangulated tori are generically 3-rigid. Using a theorem of Whiteley's [21] we show that all graphs of a class which includes the graphs of abstract triangulated closed connected surfaces are generically 3-rigid; moreover we show the generalization of this to higher dimensions.

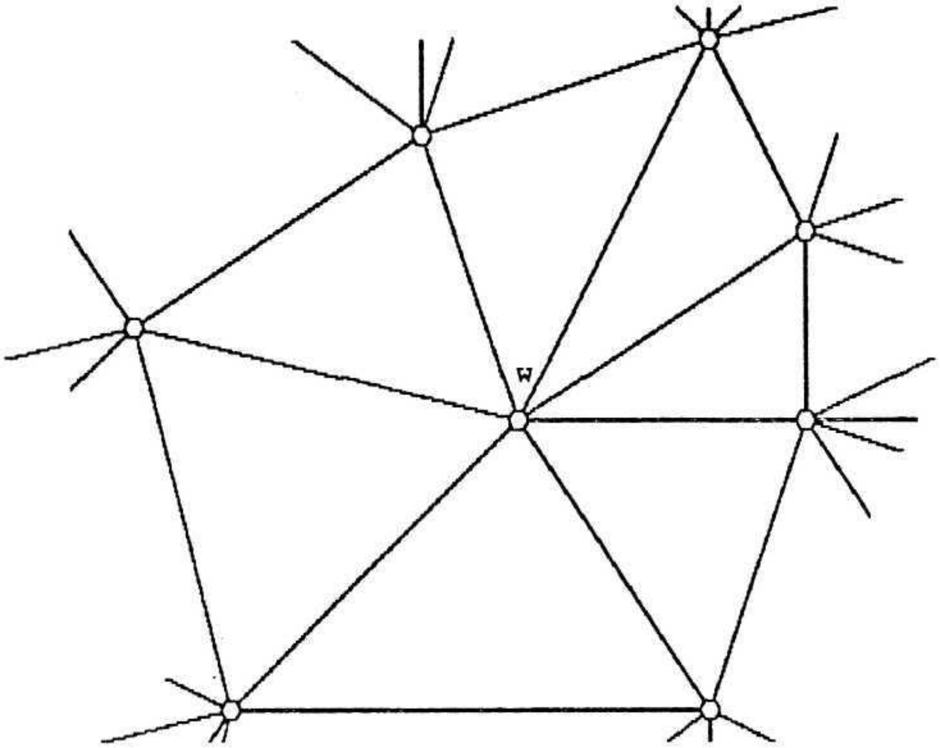
Informally, Whiteley's theorem in dimension 3 can be described as follows. Suppose we have an edge of a graph that is contained in at least two triangles. We contract this edge to form a smaller graph. The theorem says that if the contracted graph is generically 3-rigid, so is the original graph. In Figure 4, contracting the edge {u,w} in the graph B to the vertex w results in a graph A, and Whiteley's theorem says that if A is generically 3-rigid then so is B.

This is immediately applicable to triangulated polyhedral surfaces, where each edge must be in exactly two surface triangles and possibly other triangles. One can thus show that a spherical triangulated polyhedral surface is generically 3-rigid by finding a sequence of contractions that reduce it to a tetrahedron, which is well-known to be generically 3-rigid. In choosing these contractions one must avoid contracting edges which belong to more than two triangles, for contracting such an edge results in a structure that is not necessarily a surface, much less a spherical one.

For example, in Figure 5, the edge {w,x} of the polyhedral surface A can be contracted to form the tetrahedral surface B. The generic rigidity of B implies the generic rigidity of A. However, in Figure 6 we contract the edge {v,w} in A, which leaves us with C, simply two triangles joined along an edge. C is not a polyhedral surface. We left the category of triangulated polyhedral surfaces because {u,v} is in more than two triangles. C is not generically rigid and this tells us nothing about the rigidity of A.

Contracting an edge that is in more than two triangles can also lead to structures that are almost surfaces. For example, consider Figure 7. Contracting the edge {u,w} of the toroidal triangulated polyhedral surface A results in a structure that can be considered either as a "pinched torus" or as a "sphere with two edges identified."

A



B

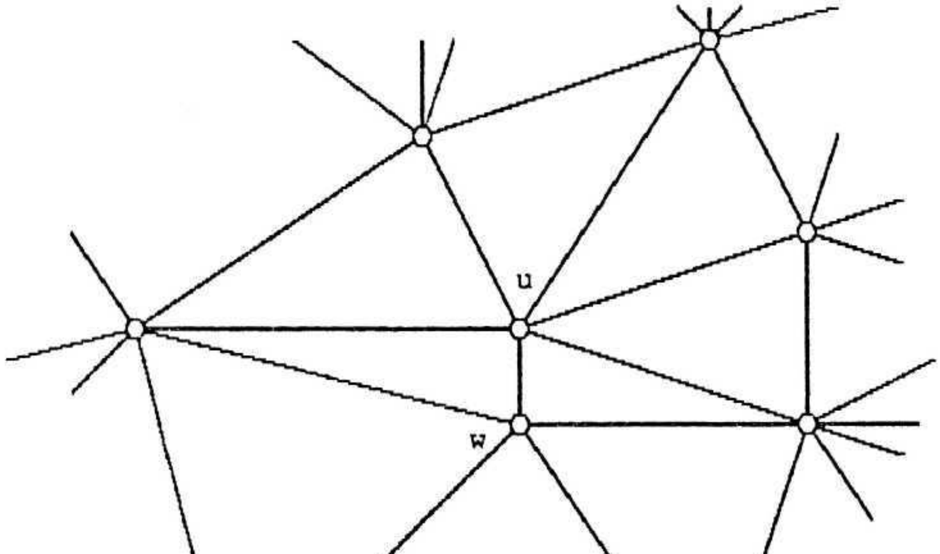


Figure 4. A contraction.

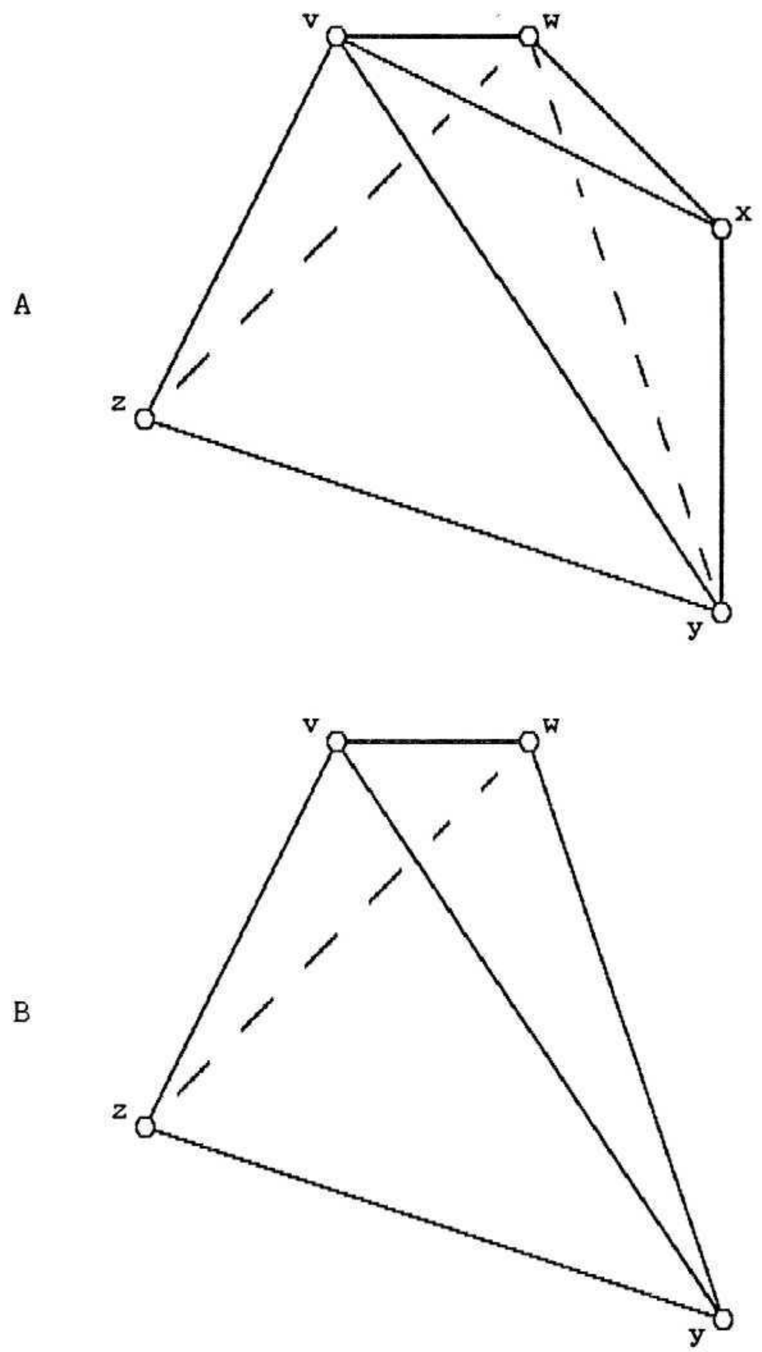


Figure 5. Contracting an edge {w,x} of a triangulated polyhedral surface.

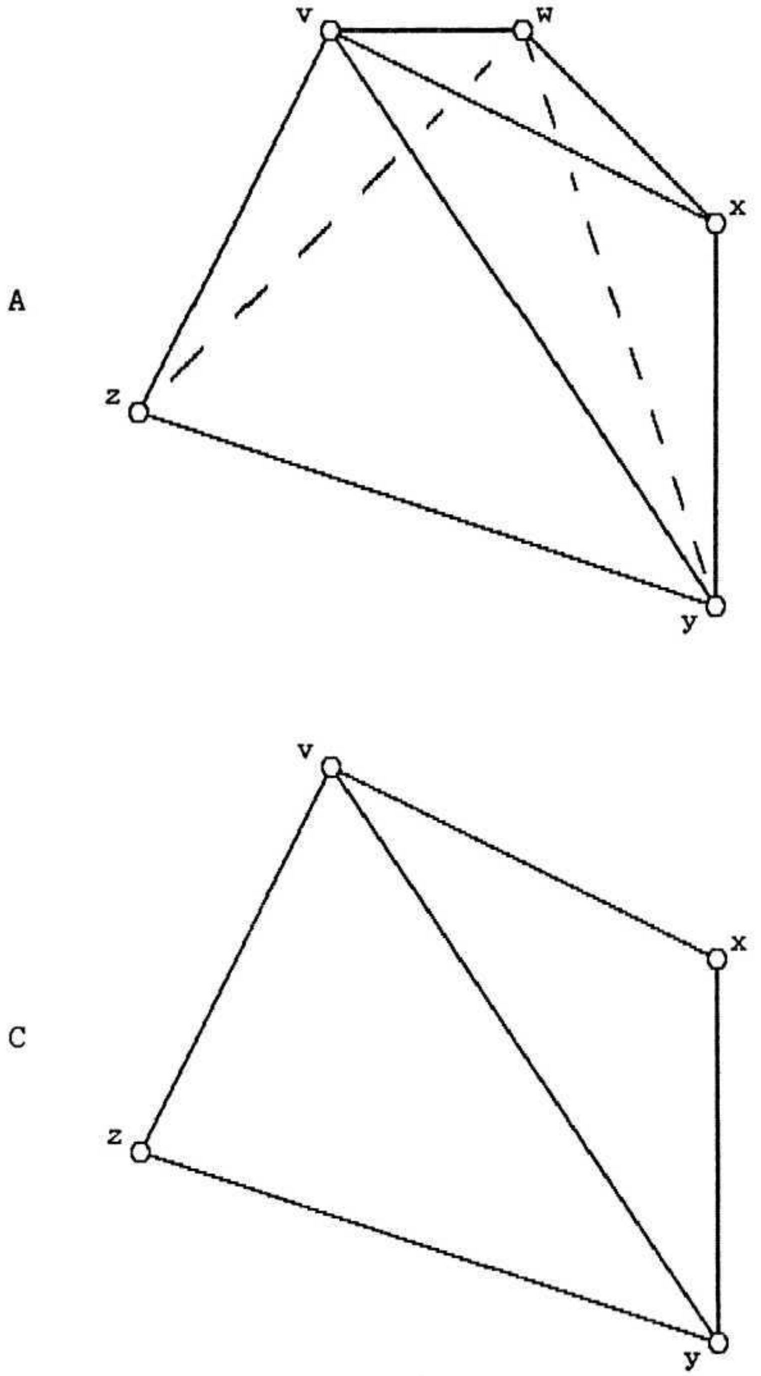
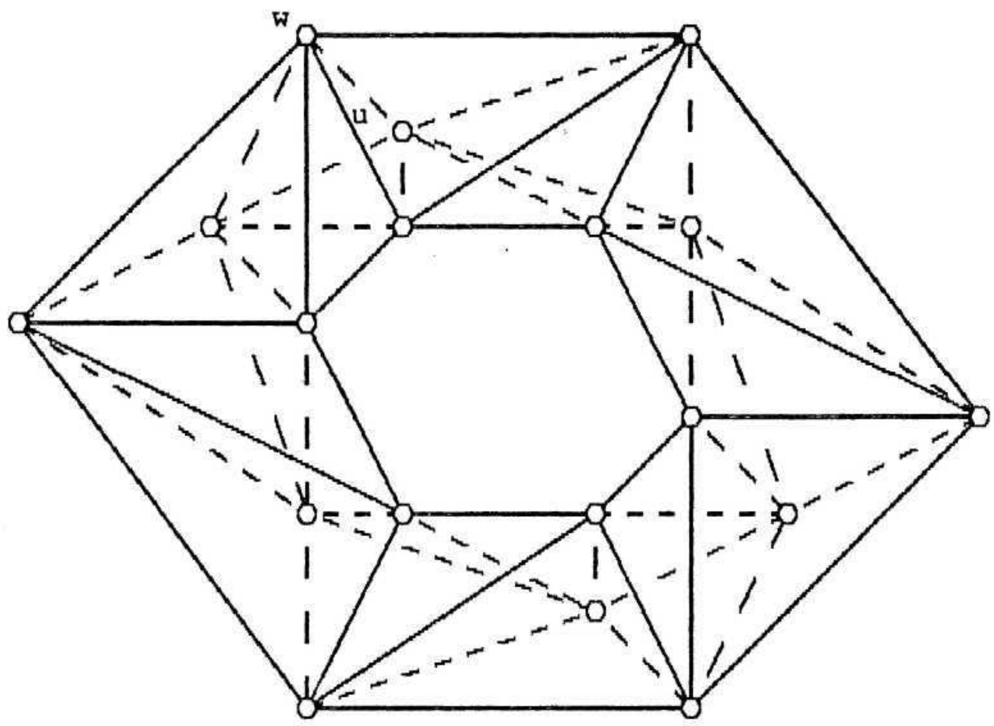


Figure 6. Contracting a short edge {v,w} of a triangulated polyhedral surface.

A



B

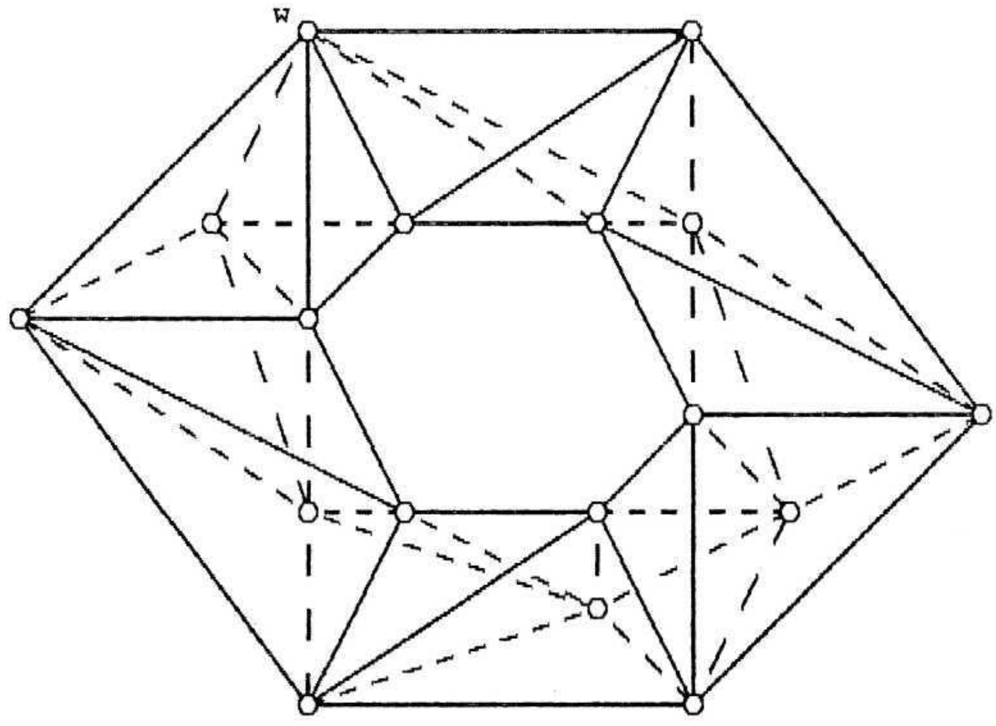


Figure 7. Contracting a short edge {u,w} of a toroidal triangulated polyhedral surface.

Whiteley calls edges of a triangulated polyhedral surface which belong to more than two triangles short; in addition, he calls a triangulated polyhedral surface short if either all its edges are short or it is a tetrahedron. Thus one can show that connected closed triangulated polyhedral surfaces of a given topological type are generically rigid by showing that all short surfaces of that type are generically 3-rigid. It is easy to show that the only short spherical triangulated polyhedral surface is the tetrahedron. Lavrenchenko [**15**] and Grünbaum and Duke [**13**] independently showed that there are twenty-one triangulated toroidal polyhedral surfaces, each of which is generically rigid. Thus we have alternative proofs of the results of Gluck [**10**] and Whiteley and Graver ([**19**] and [**11**] respectively).

The definition of short can naturally be extended to triangulations of any 2-manifold, regardless of whether or not it is embedded in 3-space. Barnette [**3**] showed that there are only two short triangulations of the projective plane, the 1-skeletons of which are generically 3-rigid. Thus altogether it has been shown that the 1-skeleton of any abstract triangulation of the sphere, the torus, or the projective plane is generically 3-rigid.

We will not attempt to lengthen this list by finding more short triangulations. Instead we carefully examine what happens when a short edge is contracted. One possible result is a "pseudo-surface" like that in Figure 7. Using methods from rigidity theory it is possible to show that, at least when there is a small number of vertices, the underlying graph of such an object is generically rigid. Along with surfaces, we would like to show that such structures are generically rigid. Their crucial characteristic in common with surfaces is that they can be considered as cycles of 2-simplices, as in elementary algebraic topology.

So we take the point of view that these triangulated polyhedral surfaces are realizations of abstract simplicial complexes. In particular we want to consider complexes that are the support complexes of 2-cycles with coefficients in $\mathbb{Z}_2$, since their realizations in 3-space include the closed triangulated polyhedral surfaces. It turns out that the properties of $\mathbb{Z}_2$ are not important and that it is easier to see what is going on if we let the coefficients be in some arbitrary abelian group. Finally, we do not want to include among the objects under consideration complexes which have realizations like the generically flexible "pseudo-surface" known as the "two bananas" in Figure 8. For this reason we introduce the concept of a minimal cycle.

Let $c(\sigma)$ denote the coefficient of an oriented simplex σ of a complex X in a chain c. If c and c′ are p-chains we say c′ is a <u>subchain</u> of c if for every oriented p-simplex σ of X either $c'(\sigma) = c(\sigma)$ or $c'(\sigma) = 0$; if c and c′ are p-cycles and c′ is a subchain of c we say c′ is a <u>subcycle</u> of c. We say a cycle c is <u>minimal</u> if its only subcycles are 0 and c.

An abstract simplicial complex X is called a <u>p-cycle complex</u> if it is the support complex of a nontrivial p-cycle. It is called a <u>minimal p-cycle complex</u> if it is the support complex of a nontrivial minimal p-cycle.

For example, consider the abstract simplicial complex X represented in Figure 9 (in which every triangle is the support complex of the boundary of a 2-simplex). The complex X is the support complex of the

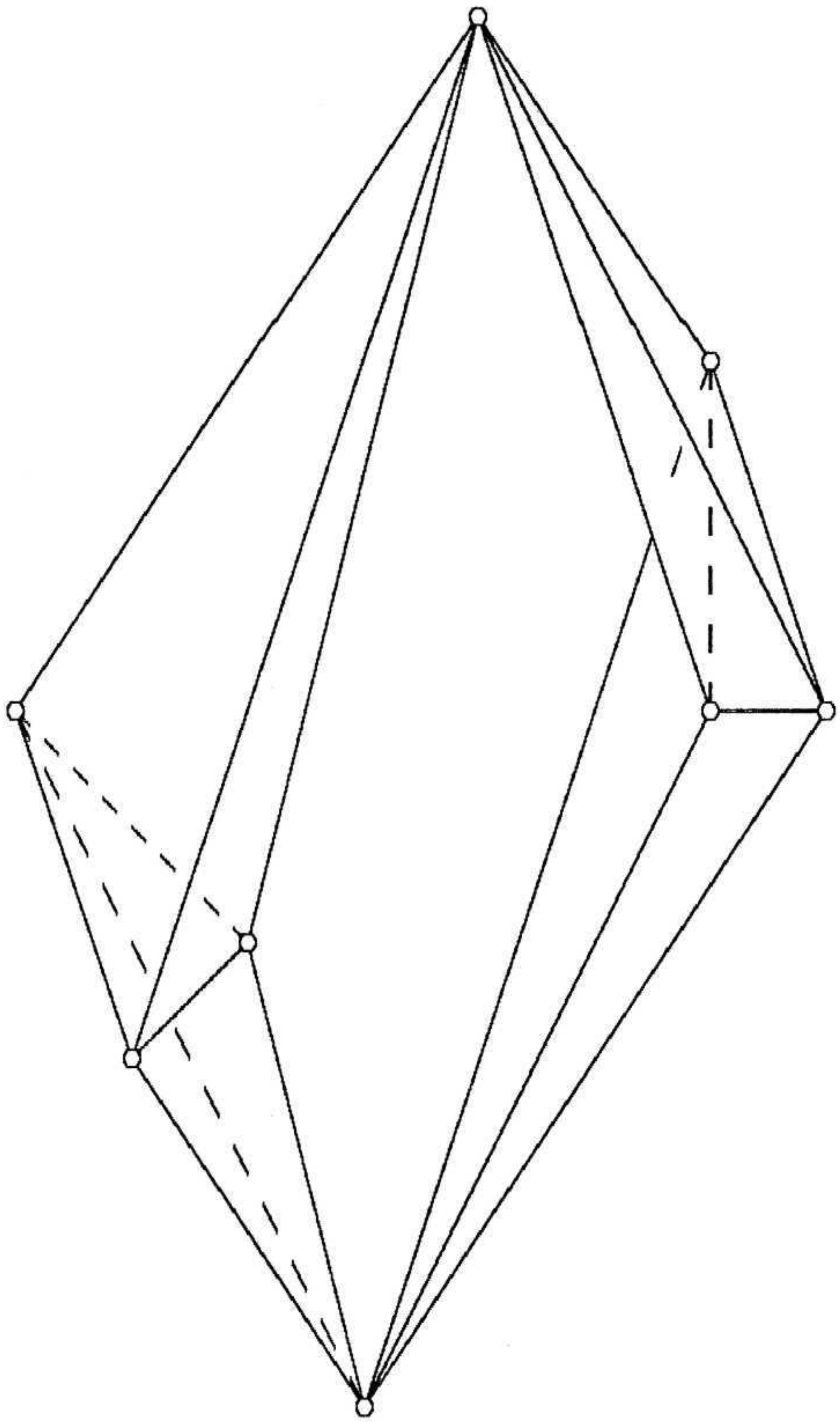


Figure 8. The "two bananas."

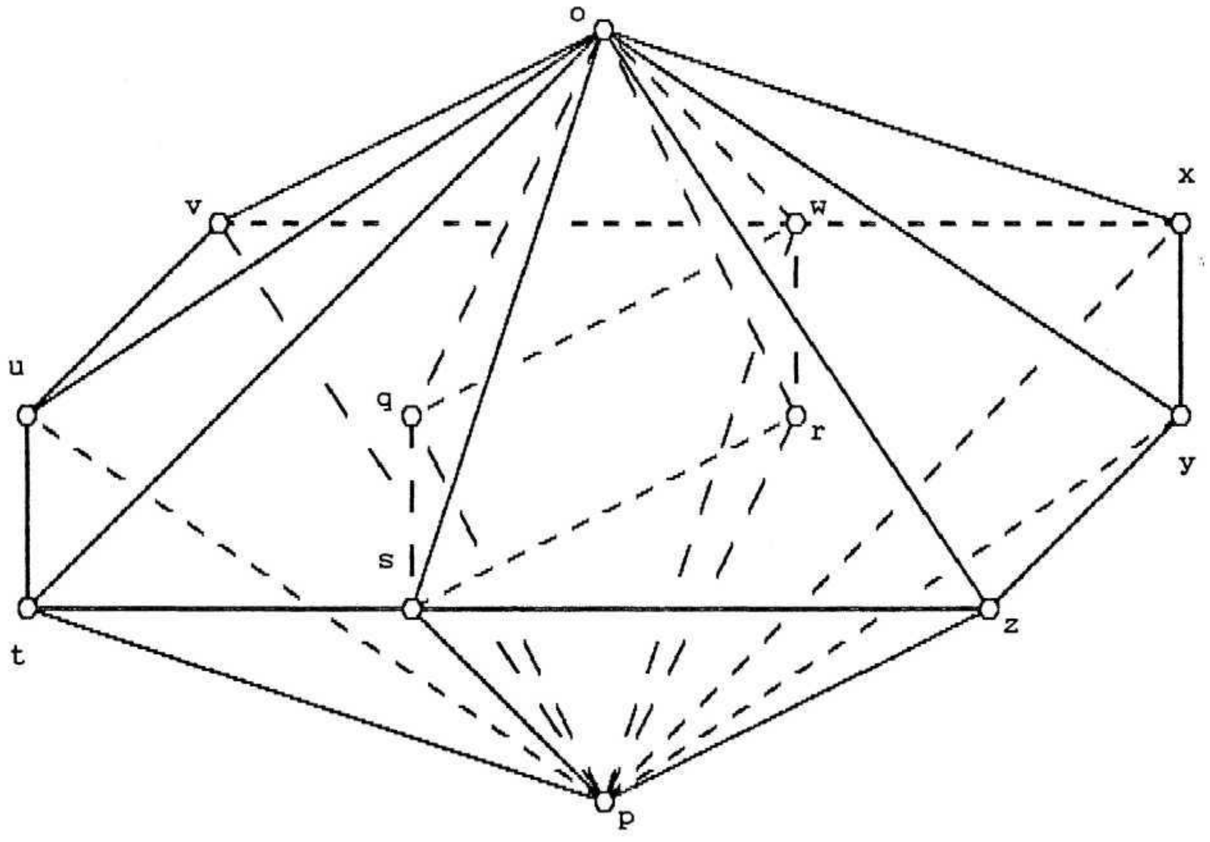


Figure 9. A minimal 2-cycle complex which contains another 2-cycle complex as a proper subcomplex.

cycle $c = [o,s,t] + [o,t,u] + [o,u,v] + [o,v,w] + 2[o,w,x] + 2[o,x,y] + 2[o,y,z] + 2[o,z,s] + 4[o,s,q] + 4[o,q,w] + 3[o,w,r] + 3[o,r,s] - [p,s,t] - [p,t,u] - [p,u,v] - [p,v,w] - 2[p,w,x] - 2[p,x,y] - 2[p,y,z] - 2[p,z,s] - 4[p,s,q] - 4[p,q,w] - 3[p,w,r] - 3[p,r,s]$ and contains the support complex of the cycle $c' = [o,s,t] + [o,t,u] + [o,u,v] + [o,v,w] + [o,w,r] + [o,r,s] - [p,s,t] - [p,t,u] - [p,u,v] - [p,v,w] - [p,w,r] - [p,r,s]$, both cycles with coefficients in $\mathbb{Z}$. One can check that c is minimal by our definition, so that X is a minimal 2-cycle complex even though supp c' is a proper subset of X.

We shall show that the realizations in 3-space of these minimal 2-cycle complexes are generically rigid. More precisely, since we are dealing with abstract complexes that may not be realizable in 3-space, we show that the 1-skeletons of such complexes are generically 3-rigid. In fact, this result generalizes to higher dimensions and we prove the following theorem.

Theorem. *The* 1-*skeleton of a minimal* (d-1)-*cycle complex,* $d \geq 3$, *is generically* d-*rigid.*

This thesis is organized as follows. Chapter 2 contains the needed definitions. In Chapter 3 is an outline and explanation of the main idea of the proof, which is to break up a minimal cycle complex into smaller rigid complexes such that their union is rigid. In Chapters 4, 5, and 6 are the proofs of the propositions mentioned in Chapter 3. Chapter 7 contains the complete detailed proof of the result. Some applications of the result are noted in Chapter 8.