2. Basic Definitions.

We consider a set to be a collection of distinct objects. We denote the order of a set S by $|S|$. $\emptyset$ denotes the empty set.

An abstract simplicial complex is a nonempty finite set X of nonempty finite sets such that if $\sigma \in X$ and τ is a nonempty subset of σ then $\tau \in X$. The sets which are elements of X are called the simplices of X. The elements of the simplices of X are called the vertices of X and we denote the set of vertices of X by V(X). If $S \subset X$ we define the vertex set of S, denoted V(S), to be $\{v \in V(X) \mid v \in \sigma \text{ for some } \sigma \in S\}$. A simplex σ of order p+1 is called a p-simplex and is said to be of dimension p; this is written $\dim \sigma = p$. The set of p-simplices of X is denoted $S^p(X)$. The dimension of X is defined to be $\max \{\dim \sigma \mid \sigma \in X\}$ and is denoted by dim X. Clearly the union of a finite number of abstract simplicial complexes of dimension $\leq d$ is an abstract simplicial complex of dimension $\leq d$.

For clarity we define the join of two simplices σ and τ, denoted $\sigma * \tau$, to be that simplex whose vertices are the union of the vertices of σ and the vertices of τ. More precisely $\sigma * \tau = \sigma \cup \tau$ because σ and τ are simply sets of vertices.

A subcomplex of X is a subset of X which is also an abstract simplicial complex. Let r be an integer. The r-skeleton of an abstract simplicial complex X is the set of all p-simplices of X with $p \leq r$; it is denoted X^r. Clearly X^r is a subcomplex of X for $r \geq 0$. We will use the fact that the union of the r-skeletons of a finite number of abstract simplicial complexes is equal to the r-skeleton of the union of the complexes.

If A is a finite set we denote the power set of A by 2^A. Clearly if A is nonempty then $2^A \setminus \{\emptyset\}$ is an abstract simplicial complex, which we call the simplex complex on A, and which we denote by $\Delta(A)$. It is also clear that if $|A| \geq 2$ then $\Delta(A)\setminus\{A\}$ is an abstract simplicial complex, which we call the simplex boundary complex on A, and which we denote by $\dot{\Delta}(A)$.

We need a precise way of describing the contraction used in Whiteley's theorem. Let X be an abstract simplicial complex. From here on we will denote the edge we want to contract by $\eta = \{u,w\}$ and we will consistently contract $\{u,w\}$ to the vertex w. Throughout this thesis we define the function q: $V(X) \to V(X)\setminus\{u\}$ by

$$q(v) = \begin{cases} w \text{ if } v = u, \\ v \text{ otherwise.} \end{cases}$$

We call q the labelling of V(X) contracting η to w. This labelling induces a function from X to the power set of $V(X)\setminus\{u\}$ which we will also denote by q and which is defined as follows: if $\sigma = \{v_0, v_1, v_2, \ldots, v_p\}$ is a p-simplex of X, set $q(\sigma)$ equal to the subset $\{q(v_0), q(v_1), q(v_2), \ldots, q(v_p)\}$ of $V(X)\setminus\{u\}$. We note that q(X) is an abstract simplicial complex, that for any p-simplex τ in q(X) there is a p-simplex σ in X such that $q(\sigma) = \tau$, and that for any integer r, $q(X)^r = q(X^r)$. We say that the abstract simplicial complex q(X) is formed from X by contracting the 1-simplex η into the vertex w. It will be clear from the context whether we take q to mean the function between the vertices of X and q(X) or the function between the simplices of X and q(X).

A graph is an abstract simplicial complex G such that dim G ≤ 1. The elements of $S^1(G)$ are called the edges of G. The set of edges of G is denoted E(G), i.e. $E(G) = S^1(G)$. A graph G is called complete if every pair of distinct vertices of G is contained in an edge of G. A subgraph G′ of G is any subcomplex of G; it is said to be spanning if it has the same vertex set as G.

We note that if $|A| \geq 3$ then the 1-skeleton of $\dot{\Delta}(A)$, the simplex boundary complex on A, is a complete graph.

We use the following basic facts from rigidity theory (see [18], [1], and [2]).

Proposition 1. *A complete graph is generically* d-*rigid for any* d.

Proposition 2. *A graph with a generically* d-*rigid spanning subgraph is generically* d-*rigid.*

Proposition 3. *The union of two generically* d-*rigid graphs with at least* d *common vertices is generically* d-*rigid.*

We also use the theorem recently proved by Walter Whiteley [21].

Theorem 4. *Let* G *be a simple graph and* $\{u,w\} = \eta \in E(G)$. *Let* q *be the labelling of* V(G) *contracting* η *to* w. *If for some integer* $d \geq 2$, q(G) *is generically* d-*rigid and* η *is an edge of at least* d-1 *distinct triangles in* G, *then* G *is also generically* d-*rigid.*

We now turn to basic definitions from elementary algebraic topology.

Let X be an abstract simplicial complex. Let σ be a simplex of X. Recall that an orientation of σ is an equivalence class of orderings of the set σ, where two orderings are equivalent if they differ by an even permutation. An <u>oriented simplex</u> is a simplex along with one of its orientations. If σ is an oriented p-simplex, $p > 0$, we let $-\sigma$ denote the simplex with the opposite orientation. If $\sigma = \{v_0, v_1, v_2, \ldots, v_p\}$ is a p-simplex we use the symbol $[v_0, v_1, v_2, \ldots, v_p]$ to denote the oriented simplex consisting of σ and the orientation which includes the ordering $(v_0, v_1, v_2, \ldots, v_p)$.

Let Γ be an abelian group. A <u>p-chain of X with coefficients in Γ</u> is a function c from the set of oriented p-simplices of X to Γ such that $c(-\sigma) = -c(\sigma)$ for every oriented p-simplex σ when $p > 0$. (This is how Hilton and Wylie defined a p-contrachain in *Homology Theory* (1960); however, the boundary operator they defined with it is the usual one associated with cochains that raises the dimension. They defined a p-chain to be a formal sum of p-simplices. We are following the approach of Munkres [**16**].) If c and c′ are p-chains we define the p-chain c+c′ by setting $(c+c')(\sigma) = c(\sigma) + c'(\sigma)$ for every oriented p-simplex σ of X. With this addition the set of p-chains becomes an abelian group which we denote by $C_p(X;\Gamma)$. If $p < 0$ or $p > \dim X$ then we let $C_p(X;\Gamma)$ denote the trivial group.

For each oriented p-simplex σ and each group element g of Γ we define an <u>elementary p-chain</u> $g\sigma$ by $g\sigma(\sigma) = g$ and $g\sigma(\tau) = 0$ if $\tau \neq \pm\sigma$. If we orient all the p-simplices of X, then we can write each p-chain uniquely as a finite sum of elementary p-chains; in this case we can consider $C_p(X;\Gamma)$ to be simply the group of formal sums of the oriented p-simplices with coefficients in G.

We say a simplex σ <u>appears</u> in a chain c if $c(\sigma)$ is nonzero when σ is oriented.

Suppose S is a subset of X. We say that a chain c is <u>carried by S</u> if every simplex appearing in c is an element of S.

Let $c \in C_p(X;\Gamma)$. Let S be the set of simplices that appear in c. Then $\bigcup_{\sigma\in S} \Delta(\sigma)$ is a subcomplex of X which we call the <u>support complex of c</u> and which we denote by supp c.

We define a homomorphism ∂_p: $C_p(X;\Gamma) \to C_{p-1}(X;\Gamma)$ by setting $\partial_p[v_0,v_1,v_2, \ldots ,v_p]$ equal to

$$\sum_{i=0}^{p}(-1)^i[v_0,v_1,v_2, \ldots ,v_{i-1},\hat{v}_i,v_{i+1}, \ldots ,v_p]$$

for each oriented p-simplex $[v_0,v_1,v_2, \ldots ,v_p]$, where the symbol $\hat{v}_i$ means that the vertex v_i is not in the simplex; if $p \leq 0$ or if $p > \dim X$ we let ∂_p be the trivial map. We call ∂_p the <u>boundary operator</u>. We define $\partial_p(g\sigma)$ to be $g(\partial_p\sigma)$. It is well-known that ∂_p is well-defined, that $\partial_p(-\sigma) = -\partial_p(\sigma)$ for every oriented p-simplex in X, and that furthermore the composition $\partial_{p-1} \circ \partial_p = 0$.

If $\rho = [v_0,v_1,v_2, \ldots ,v_p]$ is an oriented p-simplex of X and $\{u,w\} \cap \rho = \emptyset$ we denote the equivalence class of orderings $[w,v_0,v_1,v_2, \ldots ,v_p]$ by $[w,\rho]$ and $[u,w,v_0,v_1,v_2, \ldots ,v_p]$ by $[u,w,\rho]$. If ρ is not a 0-simplex then $[w,\partial\rho]$ denotes the chain $\partial\rho$ with w inserted as the leftmost element in each term of $\partial\rho$ and $[u,w,\partial\rho]$ is similarly defined. If $\rho = [v_0]$ then $[w,\partial\rho]$ denotes $[w]$ and $[u,w,\partial\rho]$ denotes $[u,w]$.

If $u \in V(X)$, suppose τ_1, τ_2, τ_3, . . . , τ_k are oriented p-simplices of X, not necessarily distinct, such that $[u,\tau_1]$, $[u,\tau_2]$, $[u,\tau_3]$, . . . , $[u,\tau_k]$ are also oriented simplices of X. Then for any elements g_1, g_2, g_3, . . . , g_k in some abelian group Γ, $\sum_{i=1}^{k} g_i\tau_i = 0$ if and only if $\sum_{i=1}^{k} g_i[u,\tau_i] = 0$.

We will use the following elementary lemma.

Lemma 5. *Suppose that* c *is a* p-*chain of an abstract simplicial complex* X *with coefficients in some abelian group* Γ. *Let* τ *be a* (p-1)-*simplex of* X. *Let* S *be the set of* p-*simplices of* X *containing* τ. *If* S *is empty then* $\partial c(\tau) = 0$. *Otherwise, if* τ *and the elements of* S *are oriented so that* τ *appears in* $\partial\sigma$ *with a + sign for each* $\sigma \in S$, *then* $\partial c(\tau) = \sum_{\sigma\in S} c(\sigma)$.

The kernel of ∂_p: $C_p(X;\Gamma) \to C_{p-1}(X;\Gamma)$ is called the group of p-cycles of X with coefficients in Γ and is denoted $Z_p(X;\Gamma)$.

Let $\{u,w\} \in S^1(X)$. Above we defined the labelling q of V(X) contracting $\{u,w\}$ to w. Recall that the chain map induced by q is a homomorphism $q_\#$: $C_p(X;\Gamma) \to C_p(q(X);\Gamma)$ defined on oriented p-simplices of X as follows: $q_\#([v_0,v_1,v_2, . . . ,v_p])$
$= [q(v_0),q(v_1),q(v_2), . . . ,q(v_p)]$ if $q(v_0)$, $q(v_1)$, $q(v_2)$, . . . , $q(v_p)$ are distinct and is trivial otherwise. It is well-known that $q_\#$ is well-defined and that $q_\#$ commutes with the boundary operator. Another elementary lemma we will use follows.

Lemma 6. *Suppose that* c *is a* p-*chain of* X *with coefficients in some abelian group* Γ. *Let* τ *be an unoriented* p-*simplex in the abstract simplicial complex* q(X). *Let* S *be the set of unoriented* p-*simplices of* X *in* $q^{-1}(\tau)$. *If* S *is empty then the coefficient of the* p-*chain* $q_\#(c)$ *on* τ, *when oriented, is* 0. *Otherwise, if* τ *and the elements of* S *are oriented so that* $q_{\#}(\sigma) = \tau$ *for each* $\sigma \in S$, *then* $(q_{\#}(c))(\tau) = \sum_{\sigma \in S} c(\sigma)$.

For completeness we repeat here the definition of minimal subchain given in Chapter 1. If c and c′ are p-chains we say c′ is a <u>subchain</u> of c if for every oriented p-simplex σ of X either $c'(\sigma) = c(\sigma)$ or $c'(\sigma) = 0$ (we say c′ is a <u>proper</u> subchain of c if $c' \neq c$); if c and c′ are p-cycles and c′ is a subchain of c we say c′ is a <u>subcycle</u> of c. We say a cycle c is <u>minimal</u> if its only subcycles are 0 and c.

X is called a <u>p-cycle complex</u> if it is the support complex of a nontrivial p-cycle. It is called a <u>minimal p-cycle complex</u> if it is the support complex of a nontrivial minimal p-cycle.

If $p > 0$ let $A = \{v_0, v_1, v_2, \ldots, v_{p+1}\}$ and consider $\dot{\Delta}(A)$, the simplex boundary complex on A, which is a subcomplex of $\Delta(A)$, the simplex complex on A. Set $\sigma = [v_0, v_1, v_2, \ldots, v_{p+1}]$. Then $\partial\sigma$ is a p-cycle on $\dot{\Delta}(A)$ with coefficients in $\mathbb{Z}$ and clearly supp $\partial\sigma = \dot{\Delta}(A)$. It is easy to show that the p-cycle complexes with the fewest vertices must be simplex boundary complexes on sets of p+2 vertices. We note that such p-cycle complexes are minimal and that their 1-skeletons are complete graphs.

To justify the remaining definitions we consider a few examples.

In Figure 10 we see that contracting the edge {u,w} of the minimal 2-cycle complex A results in two tetrahedra joined along an edge, a flexible object B. We consider the two tetrahedra C and D singly. The inverse image of C under the contraction is the complex E, which is a spherical triangulated polyhedral surface with the face {u,v,w} removed. The inverse image F of the tetrahedron D is a similar structure. The

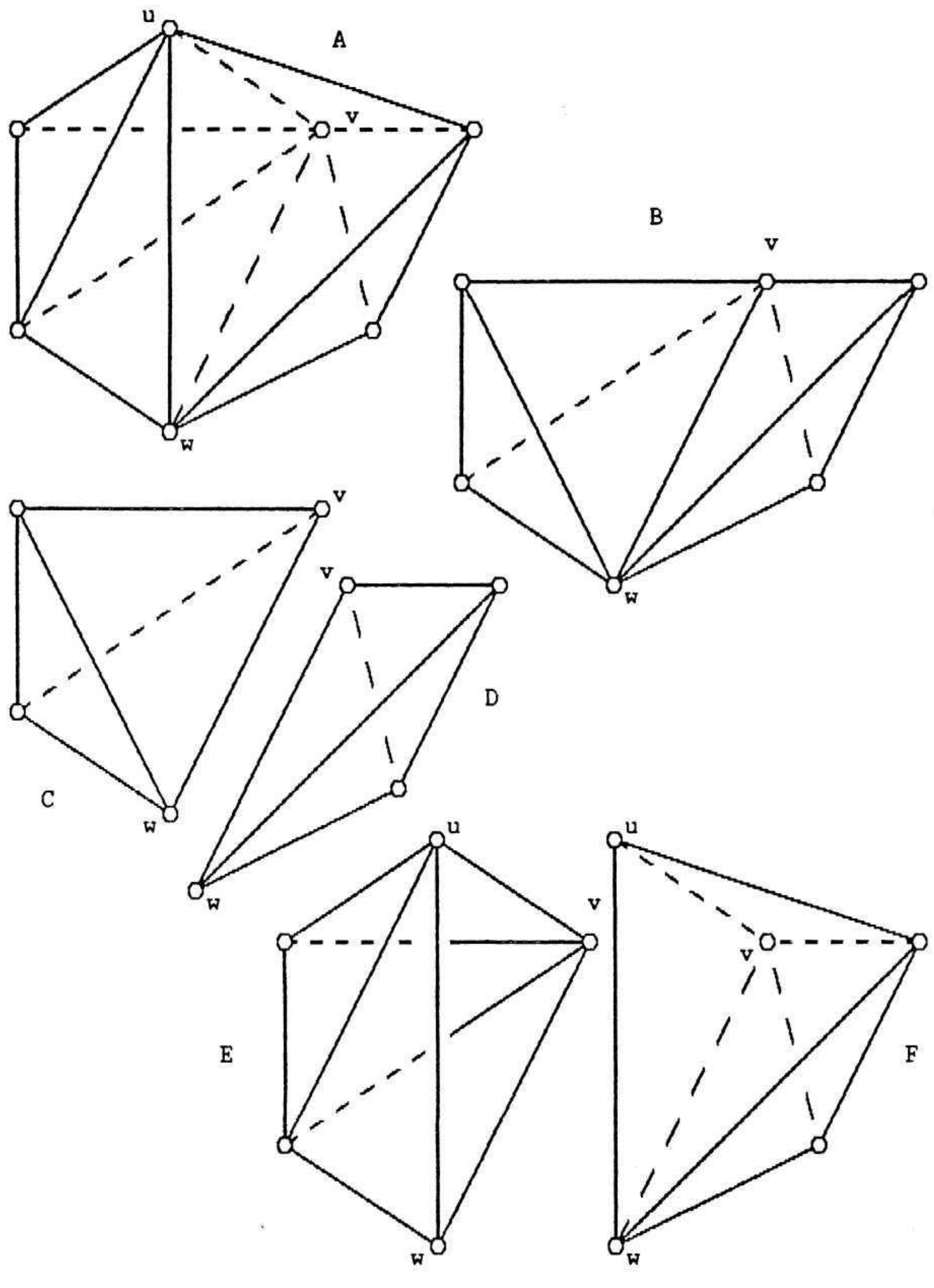

Figure 10. A decomposition induced by a contraction.

union of E and F is A. Now by Proposition 1 the tetrahedra C and D are generically 3-rigid. By Proposition 4 (Whiteley's theorem) this implies that E and F are generically 3-rigid. Then since E and F have 3 vertices in common, by Proposition 3 the complex A is generically 3-rigid.

The proof of the result follows the same lines. There are two revisions that seem to work better in the general case. One is that we find the decomposition of the cycle complex without first contracting the edge. The details of this are in Chapter 3. The other change is that we add the missing faces to the pieces we break the complex into: this keeps us working in the category of minimal cycle complexes and will later help us keep account of the pieces. Along with these revisions it is also helpful to see how a contraction classifies the simplices of a complex. Consider the 2-cycle complex X in Figure 11.

We notice that the simplicial map q which contracts the edge $\eta = \{u,w\}$ into w naturally divides the simplices of X into three disjoint classes. First of all, there are those which lose a dimension when η is contracted; these are exactly those which make up the <u>star</u> of η in X, denoted $St(\eta,X)$, i.e. those simplices which contain η. The set of the remaining simplices we call the <u>antistar</u> of η in X, denoted $Ast(\eta,X)$.

Secondly, there are pairs of simplices which are mapped together by q, not losing any dimension; this set we call the <u>suspension</u> of η in X, denoted $Susp(\eta,X)$. The suspension of η in X is the collection of simplices $\{\{u,y,z\},\{w,y,z\},\{u,v\},\{w,v\},\{u,x\},\{w,x\},\{u,y\},\{w,y\},\{u,z\},$ $\{w,z\},\{u\},\{w\}\}$. This set is somewhat harder to define in general. Recall that the <u>link</u> of a vertex u in a complex X is the set of simplices σ not containing u such that $\sigma * \{u\}$ is another simplex of X. We define

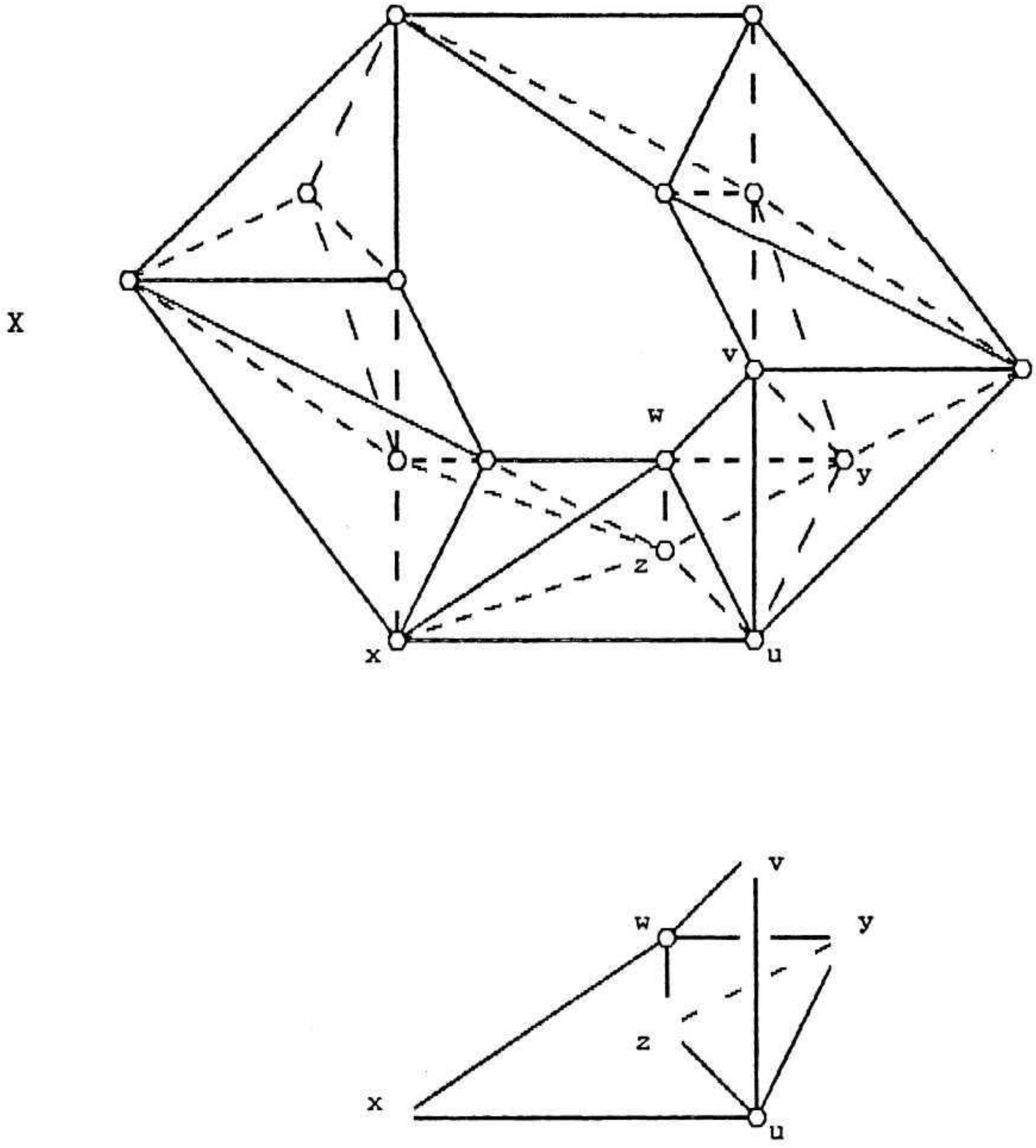

Figure 11. The suspension of the edge {u,w} in the complex X: {{u,y,z}, {w,y,z},{u,v},{w,v},{u,x},{w,x},{u,y},{w,y},{u,z},{w,z},{u},{w}}.

the <u>equator</u> of the edge $\eta = \{u,w\}$ in X to be the set of simplices that are in the link of both u and w. In Figure 11 these are $\{v\}$, $\{x\}$, $\{y\}$, $\{z\}$, and $\{y,z\}$. Finally, we define the suspension of η in X to be those simplices which are equal to either $\rho * \{u\}$ or $\rho * \{w\}$ for some ρ either in the equator of η or equal to the empty set.

Thirdly, there are those simplices which are neither in the star or the suspension of η. We call this set the <u>antisuspension</u> of η in X, denoted $\mathrm{Asusp}(\eta,X)$.

Note that the star, suspension, and antisuspension of η in X are not subcomplexes of X, although the antistar and equator are.

If $\tau \in \mathrm{Susp}(\eta,X)$ we define the <u>η-conjugate</u> of τ, denoted τ^*, to be the set $(\tau * \{u,w\})\backslash(\tau \cap \{u,w\})$, i.e. the simplex which gets mapped together with τ by q; it follows from the definitions that τ^* is a simplex in $\mathrm{Susp}(\eta,X)$. We call the pair (τ,τ^*) an <u>η-suspension pair</u>; also, we use τ_e to denote the set $\tau \cap \tau^*$, which is either the empty set or a simplex in $\mathrm{Eq}(\eta,X)$.

$\mathrm{Eq}(\eta,X)$ and $\mathrm{Susp}(\eta,X)$ have the following additional properties.

(i) If $\rho \in \mathrm{Eq}(\eta,X)$, then $\rho \cap \eta = \emptyset$, $\{\rho * \{u\}, \rho * \{w\}\} \subset \mathrm{Susp}(\eta,X)$, and $(\rho * \{u\})^* = \rho * \{w\}$.

(ii) If $\tau \in \mathrm{Susp}(\eta,X)$, then either $\tau = \tau_e * \{u\}$ and $\tau^* = \tau_e * \{w\}$, or $\tau = \tau_e * \{w\}$ and $\tau^* = \tau_e * \{u\}$. Hence $(\tau^*)^* = \tau$, $|\tau \cap \eta| = 1$, and $\mathrm{Susp}(\eta,X) \subset \mathrm{Ast}(\eta,X)$.

(iii) If $\pi \in \mathrm{St}(\eta,X)$, then $\pi\backslash\eta \in \mathrm{Eq}(\eta,X) \cup \{\emptyset\}$ and $(\pi\backslash\{u\}, \pi\backslash\{w\})$ is a η-suspension pair.

Furthermore, if q is the labelling of V(X) contracting η to w the following properties hold.

(iv) If $\tau \in \text{Susp}(\eta,X)$, then $q^{-1}(q(\tau)) \subset \{\tau,\tau^*,\tau*\tau^*\}$, and $q(\tau^*) = q(\tau)$.

(v) If $\tau \in \text{Asusp}(\eta,X)$, then $q^{-1}(q(\tau)) = \{\tau\}$.

(vi) $q(\text{Susp}(\eta,X)) \cap q(\text{Asusp}(\eta,X)) = \emptyset$.

Suppose that c is a p-chain on X. We will use the following easily proved properties.

(vii) If c is carried by $\text{St}(\eta,X)$, then ∂c is carried by $\text{St}(\eta,X) \cup \text{Susp}(\eta,X)$.

(viii) If c is carried by $\text{Ast}(\eta,X)$, then ∂c is carried by $\text{Ast}(\eta,X)$.

Finally, suppose that c is a nontrivial p-cycle, $p > 0$, and that η is a 1-simplex of supp c. It is easy to show that neither $\text{St}(\eta, \text{supp } c)$ nor $\text{Ast}(\eta, \text{supp } c)$ carries c.

Recall that in Figure 10 the complexes into which the cycle complex A decomposes are cycle complexes themselves minus a few faces. Notice that the missing face {u,v,w} would be in the star of {u,w}. Also v is in the equator of {u,w} while {u,v} and {w,v} are in the suspension of {u,w}. If $p \geq 2$, in breaking up a p-cycle complex X into subcomplexes we will be aided considerably if we can assume that for every η-suspension pair of (p-1)-simplices (τ,τ^*) (or equivalently for every (p-2)-simplex ρ in $\text{Eq}(\eta,X)$) that the corresponding p-simplex $\tau * \tau^*$ (or $\rho * \eta$) actually is a simplex (in the star of η) in X.

In general, of course, an abstract simplicial complex X will not have this property, so we add to it the necessary simplices. For $p \geq 2$ denote the collection of sets $\{\rho * \eta \mid \rho \text{ is a (p-2)-simplex in } \text{Eq}(\eta,X)\}$ by $K^p(\eta,X)$. If $K^p(\eta,X) \neq \emptyset$, then $X \cup (\bigcup_{\pi\in K^p(\eta,X)} \Delta(\pi))$ is an abstract simplicial complex, which we denote by $\tilde{X}$; otherwise we let $\tilde{X}$ denote X. In either case we call $\tilde{X}$ the <u>p-completion</u> of X over η. It is easy to show that the 1-skeletons of X and $\tilde{X}$ are the same, and hence are equally rigid.

The simplices which are added to the complex X in Figure 11 to form its 2-completion $\tilde{X}$ over η are $\{u,w,y\}$ and $\{u,w,z\}$. The needed 2-simplices $\{u,w,v\}$ and $\{u,w,x\}$ are already in X.

Note that if η is a 1-simplex of X and $\tilde{X}$ is the p-completion of X over η then $Ast(\eta,\tilde{X}) = Ast(\eta,X)$.

We will use the following properties of the p-completion $\tilde{X}$ of a complex X over a 1-simplex $\eta = \{u,w\}$.

(ix) If τ is a (p-1)-simplex in $Susp(\eta,\tilde{X})$, then $\tau * \tau^*$ is a p-simplex in $St(\eta,\tilde{X})$.

(x) If $\rho_1, \rho_2, \rho_3, \ldots, \rho_k$ are the distinct (p-2)-simplices in $Eq(\eta,\tilde{X})$, then $\eta * \rho_1, \eta * \rho_2, \eta * \rho_3, \ldots, \eta * \rho_k$ are the distinct p-simplices in $St(\eta,\tilde{X})$.

(xi) If q is the labelling of $V(\tilde{X})$ contracting η to w and τ is a (p-1)-simplex in $Susp(\eta,\tilde{X})$, then $q^{-1}(q(\tau)) = \{\tau,\tau^*,\tau*\tau^*\}$.

Now we are ready to begin considering how to break up complexes.