3. The Decomposition of a Minimal p-Cycle Complex.

Throughout this dissertation keep in mind that we have defined $\tilde{X}$ as the p-completion of X over η only when $p \geq 2$.

Let X be the minimal 2-cycle complex in Figure 11. Let c be the minimal 2-cycle on X with coefficients in $\mathbb{Z}_2$ such that X = supp c. Let c' be the restriction of c to the antistar of $\eta = \{u,w\}$ in $\tilde{X}$, i.e. $c - [u,w,v_1] - [u,w,v_2]$. We will break this chain into pieces to determine how X breaks up. $\partial c' = [u,v_1] + [w,v_1] + [u,v_2] + [w,v_2]$ is carried by $\mathrm{Susp}(\eta,\tilde{X})$. This is true in general: the chains c' and $c-c'$ are carried by $\mathrm{Ast}(\eta,\tilde{X})$ and $\mathrm{St}(\eta,\tilde{X})$ respectively. Hence $\partial c'$ is carried by $\mathrm{Ast}(\eta,\tilde{X})$ while $\partial(c-c')$ is carried by $\mathrm{St}(\eta,\tilde{X}) \cup \mathrm{Susp}(\eta,\tilde{X})$. But $\partial(c-c') = -\partial c'$ because c is a cycle. Thus $\partial c'$ must be carried by $\mathrm{Susp}(\eta,\tilde{X})$.

A <u>decomposition of c' modulo $\mathrm{Susp}(\eta,\tilde{X})$</u> is defined to be a nonempty set $D = \{c_1,c_2,c_3, \ldots ,c_k\}$ of nontrivial subchains of c' such that every simplex appearing in c' appears in exactly one subchain in D and ∂c_i is carried by $\mathrm{Susp}(\eta,\tilde{X})$ for $1 \leq i \leq k$. In fact, we need a particular kind of decomposition of c' modulo $\mathrm{Susp}(\eta,\tilde{X})$. In order that we break up X into minimal cycle complexes we require the chains c_i to be <u>minimal modulo $\mathrm{Susp}(\eta,\tilde{X})$</u>, meaning that no subchain of c_i, except c_i and 0, has its boundary chain carried by $\mathrm{Susp}(\eta,\tilde{X})$. D is said to be <u>maximal</u> if c_i is minimal modulo $\mathrm{Susp}(\eta,\tilde{X})$ for $1 \leq i \leq k$. We note that because $\partial c'$ is carried by $\mathrm{Susp}(\eta,\tilde{X})$ there exists a maximal decomposition of c' modulo $\mathrm{Susp}(\eta,\tilde{X})$.

In the case of the complex X in Figure 11 we consider the maximal decomposition $D = \{c_1,c_2,c_3\}$, where $c_3 = [u,y,z] + [w,y,z]$, c_1 is the restriction of $c' - c_3$ to the right half of Figure 11, and c_2 is the restriction of $c' - c_3$ to the left half of Figure 11.

We can now use the 2-simplices in the $\mathrm{St}(\eta,\tilde{X})$ to make cycles out of c_1, c_2, and c_3. Let $\tilde{c}_1 = c_1 + [u,w,v] + [u,w,y]$, $\tilde{c}_2 = c_2 + [u,w,x] + [u,w,z]$, and $\tilde{c}_3 = c_3 + [u,w,y] + [u,w,z]$. Figure 12 shows their support complexes. In the general case it may not be so clear how to proceed. We will need to use the following proposition, which is proved in Chapter 4.

Proposition 10. *If* c_i *is carried by* $\mathrm{Ast}(\eta,\tilde{X})$ *and* ∂c_i *is carried by* $\mathrm{Susp}(\eta,\tilde{X})$ *then there is a unique chain* b_i *carried by* $\mathrm{St}(\eta,\tilde{X})$ *such that* $\partial b_i = \partial c_i$.

We call the cycle $\tilde{c}_i = c_i - b_i$ the completion of c_i over η.

The support complexes of the completions of the c_i's are the pieces into which we break up X. We need to show that each piece is generically d-rigid.

We could show that each piece itself is a minimal cycle complex. In most cases these complexes will have fewer vertices than X and so can be assumed rigid in an induction argument. Then we would have to deal with the possibility that the number of vertices of a piece is not smaller than the number of vertices of X. We avoid this approach and instead show that either a piece is a simplex boundary complex or else it

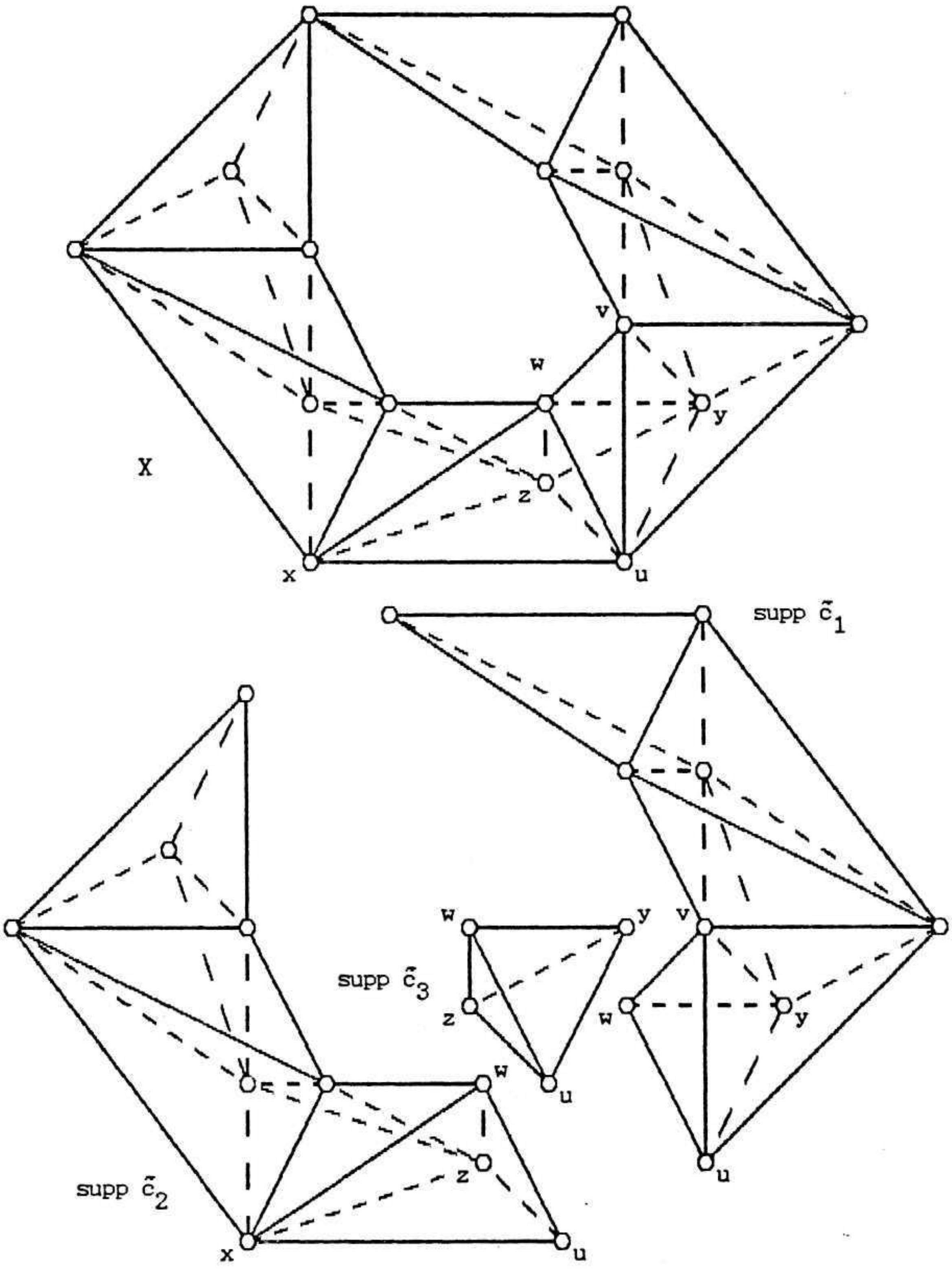

Figure 12. The support complexes of the completions of c_1, c_2, and c_3.

contracts to a minimal cycle complex which certainly has fewer vertices than X. This is Proposition 13, which is proved in Chapter 5.

Proposition 13. *If* c_i *is a* p-*chain that is minimal modulo* $\mathrm{Susp}(\eta,\tilde{X})$, *then*

(i) the chain $q_{\#}(\tilde{c}_i)$ *is a minimal* p-*cycle, and*

(ii) either the complex q(supp $\tilde{c}_i$) *equals the support complex of* $q_{\#}(\tilde{c}_i)$ *or else the support complex of* $\tilde{c}_i$ *is the simplex boundary complex* $\dot{\Delta}$(V(supp $\tilde{c}_i$)) *on its own vertex set.*

For example, suppose supp $\tilde{c}_1$, supp $\tilde{c}_2$, and supp $\tilde{c}_3$ are again the complexes in the lower half of Figure 12. Figure 13 shows the contracted complexes q(supp $\tilde{c}_1$), q(supp $\tilde{c}_2$), and q(supp $\tilde{c}_3$). The complexes q(supp $\tilde{c}_1$) and q(supp $\tilde{c}_2$) are the support complexes of the minimal 2-cycles $q_{\#}(\tilde{c}_1)$ and $q_{\#}(\tilde{c}_2)$ respectively. A similar statement is not true of q(supp $\tilde{c}_3$); however, supp $\tilde{c}_3$ is a tetrahedron.

This proposition tells us enough about the properties of the pieces to determine their rigidity. In the specific case of the complex we have just been looking at, supp $\tilde{c}_3$ is generically 3-rigid because it is a tetrahedron. Because q(supp $\tilde{c}_1$) and q(supp $\tilde{c}_2$) are simply connected surfaces they are generically 3-rigid. By Whiteley's theorem it follows that supp $\tilde{c}_1$ and supp $\tilde{c}_2$ are also generically 3-rigid. In the general case we will be working under the hypothesis that minimal (d-1)-cycle complexes with fewer vertices than X are generically d-rigid. Hence we need part (ii) of Proposition 13 to insure that if supp $\tilde{c}_i$ is not a simplex boundary complex then q(supp $\tilde{c}_i$) is a minimal (d-1)-cycle complex. Thus in any case supp $\tilde{c}_i$ is generically d-rigid.

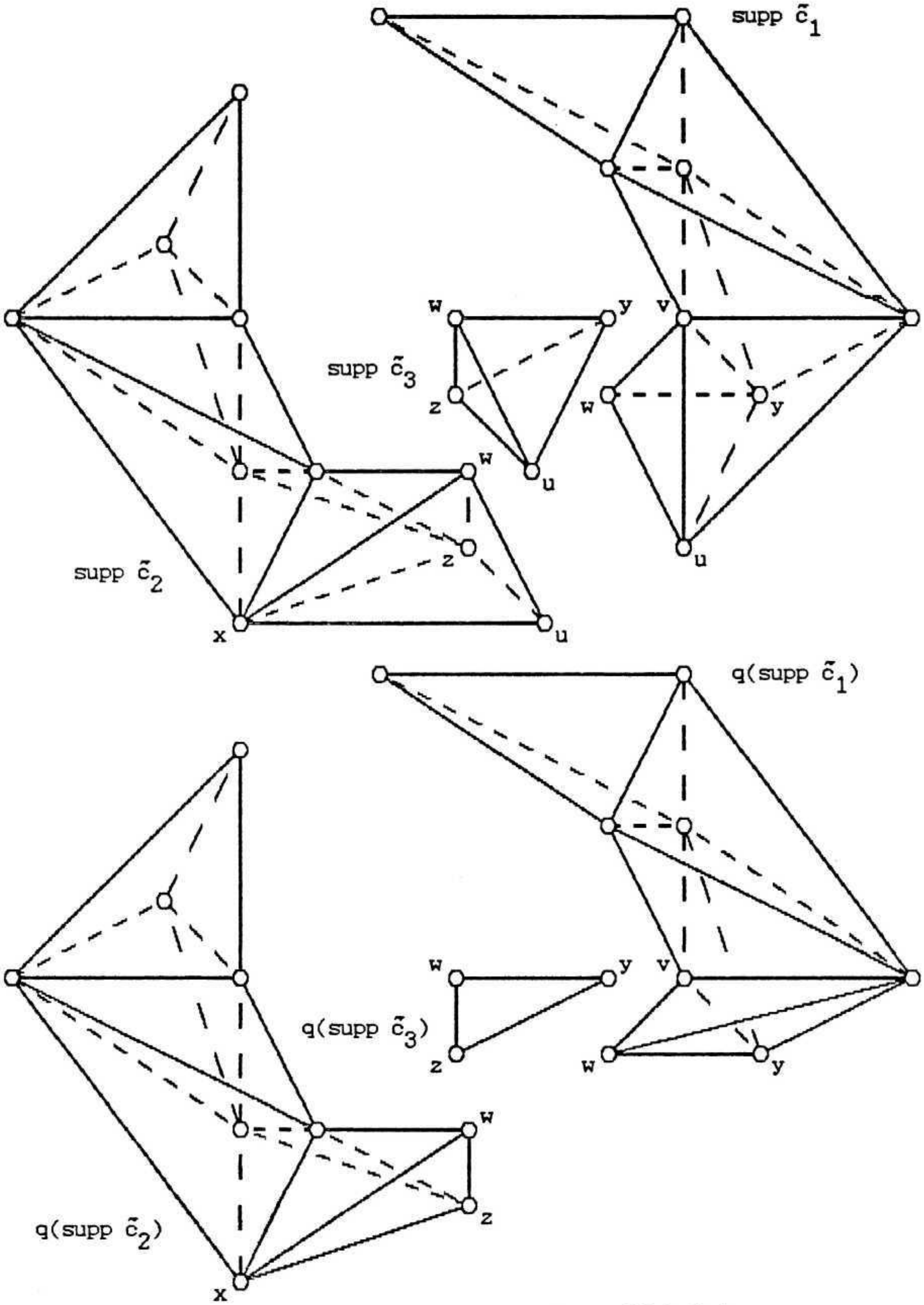

Figure 13. Contracting the complexes into which X decomposes.

It remains to show that the union of the pieces is generically d-rigid and equal to X. In our example it is easy to check that the union of these three 1-skeletons is equal to the 1-skeleton of X. Moreover, supp $\tilde{c}_1$ and supp $\tilde{c}_3$ share the vertices u, w, and y and so their union is generically 3-rigid. This union and supp $\tilde{c}_2$ have u, w, and z in common and so X is generically 3-rigid.

This sort of adding on the pieces one by one is what happens in the general case. We make the following definitions.

A finite collection C of sets is said to be <u>connected with multiplicity d</u> if for every bipartition $\{C_1,C_2\}$ of C there are sets $S_1 \in C_1$ and $S_2 \in C_2$ with at least d common points.

A <u>subgraph family</u> of a graph G is a finite collection F $= \{G_1,G_2,G_3, \ldots ,G_k\}$ of subgraphs of G. F is said to be <u>vertex connected with multiplicity d</u> if the collection of vertex sets $\{V(G_1),V(G_2),V(G_3), \ldots ,V(G_k)\}$ is connected with multiplicity d. And we say that F is <u>vertex covering</u> if $\{V(G_1),V(G_2),V(G_3), \ldots ,V(G_k)\}$ covers V(G).

We use a generalization of the basic theorem that the union of two generically d-rigid graphs with d common vertices is generically d-rigid. Proposition 14 is proved in Chapter 6.

<u>Proposition 14</u>. *If a graph* G *has a vertex covering subgraph family of generically* d-*rigid subgraphs that is vertex connected with multiplicity* d *then* G *is generically* d-*rigid.*

Thus we want to show that the family of 1-skeletons of our generically rigid pieces is a vertex covering subgraph family of the

1-skeleton of X that is vertex connected with multiplicity d. For this we have Proposition 15, which is also in Chapter 6.

Proposition 15. *If* $\{c_1, c_2, c_3, \ldots, c_k\}$ *is a decomposition of* c' *modulo* $\mathrm{Susp}(\eta, \tilde{X})$, *then* $\sum_{i=1}^{k} \tilde{c}_i = c$. *Furthermore, if* G_i *denotes the 1-skeleton of* supp $\tilde{c}_i$ *for* $1 \le i \le k$, *then* $\{G_1, G_2, G_3, \ldots, G_k\}$ *is a vertex covering subgraph family of* $\bigcup_{i=1}^{k} G_i$ *that is vertex connected with multiplicity* d.

Because $X = \mathrm{supp}\ c \subset \bigcup_{i=1}^{k} \mathrm{supp}\ \tilde{c}_i$ we are thus assured that the pieces add up to X and force X to be generically rigid.

This completes our discussion of decompositions. The remaining chapters are devoted to proving the propositions mentioned above and the result.