4. Chains and Contractions: the Proof of Proposition 10.

To prove Proposition 10 we need some facts about particular chains under contractions.

Suppose X is an abstract simplicial complex, $\{u,w\} = \eta \in S^1(X)$, and q is the labelling of V(X) contracting η to w. We say an η-suspension pair (τ,τ^*) is <u>coherently oriented</u> if $q_\#(\tau) = q_\#(\tau^*)$. The simplices τ and τ^* are coherently oriented if and only if the set of oriented simplices $\{\tau,\tau^*\}$ equals one of the following: (i) $\{[u],[w]\}$; (ii) $\{[v,u],[v,w]\}$ for some $v \in V(X)\setminus\eta$; or (iii) $\{[u,\tau_e],[w,\tau_e]\}$ for some orientation of τ_e.

<u>Proposition 7</u>. *Suppose that* c *is a* p-*chain of* X *with coefficients in some abelian group* Γ. *Then* $c \in \text{Ker } q_\#$ *if and only if*

(i) $c(\tau) = 0$ *if* τ *is an oriented* p-*simplex of* Asusp(η;X), *and*

(ii) $c(\tau) + c(\tau^*) = 0$ *if* (τ,τ^*) *is a coherently oriented* η-*suspension pair of* p-*simplices.*

Proof. Suppose $c \in \text{Ker } q_\#$. Let τ be an unoriented p-simplex in Asusp(η,X). Consider $\pi = q(\tau)$. $\dim \pi = \dim \tau$ and $q^{-1}(\pi) = \{\tau\}$. Suppose τ is oriented and orient π so that $q_\#(\tau) = \pi$. Because $c \in \text{Ker } q_\#$, $0 = (q_\#(c))(\pi) = c(\tau)$, by Lemma 6.

Let (τ,τ^*) be a coherently oriented η-suspension pair of p-simplices of X. Consider $\pi = q(\tau) = q(\tau^*)$. Now $q^{-1}(\pi) \subset \{\tau,\tau^*,\tau*\tau^*\}$ and $p = \dim \pi = \dim \tau = \dim \tau^*$, but $\dim (\tau*\tau^*) = p+1$. We can orient π so that $q_\#(\tau) = \pi$. Then because τ and τ^* are coherently oriented $q_\#(\tau^*) = \pi$. Because dim $(\tau * \tau^*) \neq p$, by Lemma 6, $c(\tau) + c(\tau^*) = (q_{\#}(c))(\pi)$, which is 0 since $c \in \text{Ker } q_{\#}$.

Conversely, suppose c satisfies conditions (i) and (ii). Let π be an oriented p-simplex in q(X). Recall that there must be a p-simplex $\tau \in X$ such that $q(\tau) = \pi$. Orient τ so that $q_{\#}(\tau) = \pi$. If $\tau \in \text{Asusp}(\eta,X)$ then $q^{-1}(\pi) = \{\tau\}$ and so by Lemma 6 $(q_{\#}(c))(\pi) = c(\tau) = 0$. If $\tau \in \text{Susp}(\eta,X)$ then $q^{-1}(\pi) \subset \{\tau,\tau^*,\tau*\tau^*\}$ and $q(\tau^*) = \pi$; because dim $\tau = p$ = dim τ^* and dim $(\tau*\tau^*) \neq p$, by Lemma 6 $(q_{\#}(c))(\pi) = c(\tau) + c(\tau^*)$ if τ^* is also oriented so that $q_{\#}(\tau^*) = \pi$. In this case τ and τ^* are coherently oriented and so $(q_{\#}(c))(\pi) = c(\tau) + c(\tau^*) = 0$. Finally, if $\tau \in \text{St}(\eta,X)$ then dim π = dim $\tau - 1$, contradicting our choice of τ. Hence for any oriented p-simplex π in q(X) we have $(q_{\#}(c))(\pi) = 0$. Thus $q_{\#}(c) = 0$. Q.E.D.

<u>Proposition</u> <u>8</u>. *For* $p > 0$ *suppose that* c *is a* p-*cycle of* X *with coefficients in some abelian group* Γ. *If* c *is carried by* $\text{St}(\eta,X) \cup \text{Susp}(\eta,X)$, *then* $q_{\#}(c) = 0$.

Proof. Suppose τ is an oriented p-simplex of Asusp(η,X). Because c is carried by $\text{St}(\eta,X) \cup \text{Susp}(\eta,X)$ we have $c(\tau) = 0$.

Suppose τ is an oriented p-simplex in Susp(η,X). Orient τ^* so that τ and τ^* are coherently oriented. Because $p > 0$, $\{\tau,\tau^*\} \neq \{[u],[w]\}$.

If $\{\tau,\tau^*\} = \{[u,\tau_e],[w,\tau_e]\}$ for some orientation of τ_e, then consider $\partial c(\tau_e)$. τ_e appears in $\partial[u,\tau_e]$ and $\partial[w,\tau_e]$ with + signs. Any other unoriented p-simplex ρ containing τ_e contains neither u nor w and so is not in $\text{St}(\eta,X) \cup \text{Susp}(\eta,X)$; hence for either orientation of ρ, $c(\rho) = 0$ by hypothesis. So by Lemma 5 $\partial c(\tau_e) = c([u,\tau_e]) + c([w,\tau_e])$. Because c is a cycle we have $0 = c(\tau) + c(\tau^*)$.

Otherwise $\{\tau,\tau^*\} = \{[v,u],[v,w]\}$ for some $v \in V(X)\backslash\eta$ and so $\tau_e = [v]$. By an argument similar to that used above we still have $0 = c(\tau) + c(\tau^*)$.

Thus $c \in \text{Ker } q_\#$ by Proposition 7. Q.E.D.

Proposition 9. *Let* $\tilde{X}$ *be the* p-*completion of* X *over* η. *Suppose that* c *is a* (p-1)-*cycle of* $\tilde{X}$ *with coefficients in some abelian group* Γ, *and that* c *is carried by* $\text{Susp}(\eta,\tilde{X})$.

(i) If $\rho_1, \rho_2, \rho_3, \ldots, \rho_k$ *are the distinct* (p-2)-*simplices in* $\text{Eq}(\eta,\tilde{X})$, *arbitrarily oriented, then there exist unique* $g_1, g_2, g_3, \ldots, g_k$ *in* Γ *such that* $c = \sum_{i=1}^{k} g_i([w,\rho_i]-[u,\rho_i])$.

(ii) There is a unique p-*chain* b *carried by* $\text{St}(\eta,\tilde{X})$ *such that* $\partial b = c$. *It is* $b = \sum_{i=1}^{k} g_i[u,w,\rho_i]$.

Proof. (i) For $1 \leq i \leq k$, because $\rho_i \in \text{Eq}(\eta,\tilde{X})$ we have that the pair of oriented (p-1)-simplices $\tau_i = [u,\rho_i]$ and $\tau_i^* = [w,\rho_i]$ is a coherently oriented η-suspension pair. By the definition of the suspension of η in $\tilde{X}$, $(\tau_1,\tau_1^*), (\tau_2,\tau_2^*), (\tau_3,\tau_3^*), \ldots, (\tau_k,\tau_k^*)$ must be exactly the distinct η-suspension pairs of (p-1)-simplices in $\tilde{X}$. Because c is a (p-1)-cycle carried by $\text{Susp}(\eta,\tilde{X})$, by Proposition 8 $q_\#(c) = 0$ and so by Proposition 7 $c(\tau_i) = -c(\tau_i^*)$ for $1 \leq i \leq k$. Setting $g_i = c(\tau_i^*)$ for $1 \leq i \leq k$ gives us $c = \sum_{i=1}^{k} g_i([w,\rho_i]-[u,\rho_i])$ because c is carried by the the (p-1)-simplices of $\text{Susp}(\eta,\tilde{X})$. Because the suspension pairs $(\tau_1,\tau_1^*), (\tau_2,\tau_2^*), (\tau_3,\tau_3^*), \ldots, (\tau_k,\tau_k^*)$ are distinct $g_1, g_2, g_3, \ldots, g_k$ are unique.

(ii) Because c is a cycle $0 = \partial c = \sum_{i=1}^{k} g_i(\rho_i - [w,\partial\rho_i] - \rho_i + [u,\partial\rho_i])$ $= \sum_{i=1}^{k} g_i([u,\partial\rho_i] - [w,\partial\rho_i])$. Because w does not appear in $[u,\partial\rho_i]$ for any i, this implies $\sum_{i=1}^{k} g_i[w,\partial\rho_i] = 0$. By the definition of $\tilde{X}$, $[u,w,\rho_i]$ is an oriented p-simplex of $\tilde{X}$ for $1 \le i \le k$. Hence each oriented p-simplex appearing in the chain $[u,w,\partial\rho_i]$ actually is in X. So $\sum_{i=1}^{k} g_i[w,\partial\rho_i] = 0$ implies $\sum_{i=1}^{k} g_i[u,w,\partial\rho_i] = 0$.

Consider $b = \sum_{i=1}^{k} g_i[u,w,\rho_i]$. By the definition of $\tilde{X}$, as noted above, b is indeed a p-chain on $\tilde{X}$ and is clearly carried by $St(\eta,\tilde{X})$. Furthermore $\partial b = \sum_{i=1}^{k} g_i([w,\rho_i] - [u,\rho_i] + [u,w,\partial\rho_i]) = c + 0 = c$.

Conversely, suppose that b is a p-chain carried by $St(\eta,\tilde{X})$ such that $\partial b = c$. Recall that the distinct p-simplices in $St(\eta,\tilde{X})$ are precisely $\eta * \rho_1, \eta * \rho_2, \eta * \rho_3, \ldots, \eta * \rho_k$ by the definition of $\tilde{X}$. Hence b $= \sum_{i=1}^{k} h_i[u,w,\rho_i]$ for some $h_1, h_2, h_3, \ldots, h_k$ in Γ. Then $\sum_{i=1}^{k} g_i([w,\rho_i] - [u,\rho_i]) = c = \partial b = \sum_{i=1}^{k} h_i([w,\rho_i] - [u,\rho_i] + [u,w,\partial\rho_i])$. For $1 \le i \le k$, because $\eta \cap \rho_i = \emptyset$ neither of the terms $[w,\rho_i]$ or $[u,\rho_i]$ can be either orientation of any the terms containing both u and w. Then because $\rho_1, \rho_2, \rho_3, \ldots, \rho_k$ are distinct we must have $g_i = h_i$ for $1 \le i \le k$. Thus $b = \sum_{i=1}^{k} g_i[u,w,\rho_i]$. Q.E.D.

<u>Proposition</u> <u>10</u>. *Suppose* c_i *is a* p-*chain of* $\tilde{X}$ *with coefficients in some abelian group* Γ. *If* c_i *is carried by* $Ast(\eta,\tilde{X})$ *and* ∂c_i *is carried by*

$\mathrm{Susp}(\eta,\tilde{X})$ *then there is a unique chain* b_i *carried by* $\mathrm{St}(\eta,\tilde{X})$ *such that* $\partial b_i = \partial c_i$.

Proof. Because ∂c_i is a (p-1)-cycle, Proposition 10 directly follows from Proposition 9. Q.E.D.

5. The Difference Between $q(\text{supp } \tilde{c}_i)$ and supp $q_\#(\tilde{c}_i)$: the Proof of Proposition 13.

One technical problem we encounter is that a contraction, as used in Whiteley's theorem, is a simplicial map, while our category of minimal cycle complexes is defined by chains. Recall that one of the pieces into which we broke up the example complex in Chapter 3 is a tetrahedron, which q maps to a 2-simplex while the chain map $q_\#$ maps to 0. This is essentially the only difference between q and $q_\#$ and we show this in this chapter.

Suppose X is an abstract simplicial complex with a 1-simple $\eta = \{u,w\}$, and let q be the labelling of V(X) contracting η to w. Lemma 11 simply says that if c is a p-chain of X then supp $q_\#(c)$ is always a subcomplex of q(supp c), and that if they differ by a p-simplex then this p-simplex must be part of a set of p-simplices which form a subchain of c that $q_\#$ collapses to 0. We do not need the special properties of q to prove this: it is true of any simplicial map.

<u>Lemma</u> <u>11</u>. *(i) If* c *is a* p*-chain of* X *with coefficients in some abelian group* Γ*, then* supp $q_\#(c)$ *is a subcomplex of* q(supp c), *i.e. the support of the image under* $q_\#$ *of the chain* c *is a subcomplex of the image under* q *of the support of* c. *Clearly both complexes are subcomplexes of* q(X).

(ii) If τ *is a* p*-simplex in* q(supp c) *that is not in* supp $q_\#(c)$, *and if* S' *is the set of distinct* p*-simplices of* supp c *in* $q^{-1}(\tau)$, *then there*

are at least 2 distinct p-*simplices in* S', *and for any orientation of* τ *and the elements of* S' *such that* $q_{\#}(\sigma) = \tau$ *whenever* $\sigma \in S'$, *we have that* $c(\sigma) \neq 0$ *if* $\sigma \in S'$ *and that* $\sum_{\sigma \in S'} c(\sigma) = 0$.

Proof. (i) Let $\rho \in \operatorname{supp} q_{\#}(c)$. Then ρ is a nonempty subset of some p-simplex τ in $q(X)$ such that $(q_{\#}(c))(\tau) \neq 0$. Let T be the set of p-simplices of X in $q^{-1}(\tau)$. By Lemma 6 since $(q_{\#}(c))(\tau) \neq 0$ we have that T is nonempty; furthermore, let us orient τ and the elements of T so that $q_{\#}(\sigma) = \tau$ for each $\sigma \in T$: then $\sum_{\sigma \in T} c(\sigma) = (q_{\#}(c))(\tau) \neq 0$. Hence there is an oriented p-simplex σ in X such that $q_{\#}(\sigma) = \tau$ and $c(\sigma) \neq 0$. Because $c(\sigma) \neq 0$ we have $\sigma \in \operatorname{supp} c$, implying $\tau \in q(\operatorname{supp} c)$. Because $q(\operatorname{supp} c)$ is an abstract simplicial complex and $\emptyset \neq \rho \subset \tau \in q(\operatorname{supp} c)$, we have $\rho \in q(\operatorname{supp} c)$. Thus $\operatorname{supp} q_{\#}(c) \subset q(\operatorname{supp} c)$.

(ii) Suppose that τ is a p-simplex in $q(\operatorname{supp} c)$ that is not in $\operatorname{supp} q_{\#}(c)$. By the definition of $q(\operatorname{supp} c)$ there is a simplex $\rho \in \operatorname{supp} c$ such that $q(\rho) = \tau$. In fact there must be a p-simplex $\sigma \subset \rho$ such that $q(\sigma) = \tau$. So S' is nonempty. Clearly S' is a subset of S, the set of p-simplices of X in $q^{-1}(\tau)$. Orient τ and the elements of S so that $q_{\#}(\sigma) = \tau$ for each $\sigma \in S$. By Lemma 6 $\sum_{\sigma \in S} c(\sigma) = (q_{\#}(c))(\tau)$. Because τ is a p-simplex of $q(X)$ that is not in $\operatorname{supp} q_{\#}(c)$ it follows that $0 = (q_{\#}(c))(\tau) = \sum_{\sigma \in S} c(\sigma)$. If $\sigma \in S \setminus S'$ then σ is not in supp c, and so $c(\sigma) = 0$. Hence $\sum_{\sigma \in S'} c(\sigma) = 0$. Finally, since the p-simplices in S' are all in supp c it follows that c in nonzero on all of them. Because S' is nonempty this forces it to contain at least 2 simplices. Q.E.D.

We will want to use the second part of Lemma 11 to show exactly how $q(\text{supp } \tilde{c}_i)$ and supp $q_{\#}(\tilde{c}_i)$ differ when $\tilde{c}_i$ is one of the completed p-cycles which form the basis of our decomposition of a cycle complex. Hence we need to show that when c is a p-cycle and q(supp c) $\neq$ supp $q_{\#}(c)$, then $q(\text{supp } \tilde{c}_i)$ and supp $q_{\#}(\tilde{c}_i)$ do differ by a p-simplex. This does depend on the properties of q.

Lemma 12. *Suppose that* c *is a* p-*cycle of* X *with coefficients in some abelian group* Γ. *If* supp $q_{\#}(c) \neq q(\text{supp } c)$, *then there is a* p-*simplex in* q(supp c) *that is not in* supp $q_{\#}(c)$.

Proof. If supp $q_{\#}(c) \neq q(\text{supp } c)$, then by the first part of Lemma 11 there is some simplex τ which is in q(supp c) but not in supp $q_{\#}(c)$. Because τ is in q(supp c) there is a simplex $\psi \in$ supp c such that $q(\psi) = \tau$. By the definition of supp c there must be a p-simplex $\pi \in \text{supp } c \subset X$ such that π contains ψ and $c(\pi) \neq 0$ when π is oriented. It is clear that because $\psi \subset \pi$ we have $\tau = q(\psi) \subset q(\pi)$, so because τ is not in supp $q_{\#}(c)$ neither is $q(\pi)$. Since $q(\pi)$ is in q(supp c), if it is a p-simplex we are done. Otherwise $\dim q(\pi) < p$ and hence π must be in $St(\eta,X)$ (and p must be positive); we assume this for the remainder of the proof.

Orient π so that $\pi = [u,w,\rho]$ for some oriented (p-2)-simplex ρ (if p = 1 the argument for $\pi = [u,w]$ is similar to what follows). Consider the oriented (p-1)-simplex $[w,\rho]$. Let S be the set of p-simplices of X containing $[w,\rho]$. S is nonempty because $\pi \in S$. Hence if the elements of S are oriented so that $[w,\rho]$ appears in $\partial\sigma$ with a + sign for each $\sigma \in S$, then $\partial c([w,\rho]) = \sum_{\sigma \in S} c(\sigma)$. Because c is a cycle $0 = \partial c([w,\rho]) = \sum_{\sigma \in S} c(\sigma)$.

But $c(\pi) \neq 0$. Hence $[w,\rho]$ also appears in $\partial\sigma$ with a + sign for some oriented p-simplex $\sigma \neq \pi$ such that $c(\sigma) \neq 0$. Because $c(\sigma) \neq 0$, $\sigma \in$ supp c. Because $[w,\rho]$ appears in $\partial\sigma$ with a + sign σ can be written as $[v,w,\rho]$ for some vertex $v \neq u$.

Recall that $q(\pi) = \rho * \{w\}$ is not in supp $q_{\#}(c)$. Now $q(\sigma) = \sigma \supset q(\pi)$ so therefore σ also is in q(supp c) but not in supp $q_{\#}(c)$. Because σ is a p-simplex we are done. Q.E.D.

We are now ready to prove Proposition 13.

Proposition 13. *Let* $\tilde{X}$ *be the completion of* X *over* η *for some* $p \geq 2$. *Suppose that* c_i *is a* p*-chain on* $\tilde{X}$ *with coefficients in some abelian group* Γ *and that* c_i *is carried by* $\mathrm{Ast}(\eta,\tilde{X})$ *and that* ∂c_i *is carried by* $\mathrm{Susp}(\eta,\tilde{X})$. *If* c_i *is minimal modulo* $\mathrm{Susp}(\eta,\tilde{X})$, *then*

(i) the chain $q_{\#}(\tilde{c}_i)$ *is a minimal* p*-cycle on the complex* q(X), *and*

(ii) either the complex q(supp $\tilde{c}_i$) *equals the support complex of* $q_{\#}(\tilde{c}_i)$ *or else the support complex of* $\tilde{c}_i$ *is the simplex boundary complex* $\dot{\Delta}(V(\text{supp } \tilde{c}_i))$ *on its own vertex set.*

Proof. (i) $\partial(q_{\#}(\tilde{c}_i)) = q_{\#}(\partial\tilde{c}_i) = q_{\#}(0) = 0$. To show that $q_{\#}(\tilde{c}_i)$ is minimal let s′ be a subcycle of $q_{\#}(\tilde{c}_i)$ and define a p-chain s on X by

$$s(\sigma) = \begin{cases} \tilde{c}_i(\sigma) & \text{if } q_{\#}(\sigma) \neq 0 \text{ and } s'(q_{\#}(\sigma)) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The proof that s is a subchain of $\tilde{c}_i$ such that $q_{\#}(s) = s'$ is straightforward.

Suppose σ is an oriented p-simplex in $\mathrm{St}(\eta,\tilde{X})$. Then dim $q(\sigma) < p$ so $q_{\#}(\sigma) = 0$. By definition $s(\sigma) = 0$. Thus s is carried by $\mathrm{Ast}(\eta,\tilde{X})$. Because $\tilde{c}_i$ and c_i are identical on $\mathrm{Ast}(\eta,\tilde{X})$ it follows that s is a subchain of c_i.

We have $q_{\#}(\partial s) = \partial(q_{\#}(s)) = \partial s' = 0$. Hence by Proposition 7, ∂s is trivial on $\mathrm{Asusp}(\eta,\tilde{X})$, i.e. is carried by $\mathrm{St}(\eta,\tilde{X}) \cup \mathrm{Susp}(\eta,\tilde{X})$. But because s is carried by $\mathrm{Ast}(\eta,\tilde{X})$, ∂s is carried by $\mathrm{Ast}(\eta,\tilde{X})$. Therefore ∂s is carried by $\mathrm{Susp}(\eta,\tilde{X})$.

Hence because c_i is minimal modulo $\mathrm{Susp}(\eta,\tilde{X})$ either $s = 0$ or $s = c_i$. Clearly $s = 0$ implies that $s' = q_{\#}(s) = 0$. If $s = c_i$ then $s' = q_{\#}(c_i) = q_{\#}(\tilde{c}_i)$. Hence $q_{\#}(\tilde{c}_i)$ is minimal.

(ii) Suppose that $\mathrm{supp}\ q_{\#}(\tilde{c}_i) \neq q(\mathrm{supp}\ \tilde{c}_i)$. By Lemma 12 there is a p-simplex τ in $q(\mathrm{supp}\ c_i)$ that is not in $\mathrm{supp}\ q_{\#}(c_i)$. Then by Lemma 11, if S' is the set of distinct p-simplices of $\mathrm{supp}\ \tilde{c}_i$ in $q^{-1}(\tau)$, then there are at least 2 distinct p-simplices in S', and for any orientation of τ and the elements of S' such that $q_{\#}(\sigma) = \tau$ whenever $\sigma \in S'$, we have that $c_i(\sigma) \neq 0$ if $\sigma \in S'$ and that $\sum_{\sigma\in S'} c_i(\sigma) = 0$. If $\sigma \in S'$ the dimension of σ does not decrease under the contraction q so σ must be in $\mathrm{Ast}(\eta,\tilde{X})$; furthermore, because there are at least 2 distinct p-simplices in $q^{-1}(q(\sigma))$, σ cannot be in $\mathrm{Asusp}(\eta,\tilde{X})$: thus σ must be in $\mathrm{Susp}(\eta,\tilde{X})$ and so the p-simplices in $q^{-1}(q(\sigma))$ must be exactly σ and σ^*. Hence $w \in \tau$. Orient τ. We defined p-complete only when $p \geq 2$, so there is an oriented (p-1)-simplex ρ such that $\tau = [w,\rho]$. Then $\{\sigma,\sigma^*\} = \{[u,\rho],[w,\rho]\}$.

Without loss of generality let $\sigma = [w,\rho]$ and $\sigma^* = [u,\rho]$. Let $g = c_i(\sigma)$ and $g^* = c_i(\sigma^*)$. Recall that $c_i(\sigma) + c_i(\sigma^*) = 0$ so that $g^* = -g$. We have that $g\sigma + g^*\sigma^*$ is a subchain of c_i and that $\partial(g\sigma + g^*\sigma^*) = \partial(g\sigma - g\sigma^*) = g(\rho - [w,\partial\rho] - \rho + [u,\partial\rho]) = g([u,\partial\rho] - [w,\partial\rho])$, which is carried by $\mathrm{Susp}(\eta,\tilde{X})$. By hypothesis c_i is minimal modulo $\mathrm{Susp}(\eta,\tilde{X})$ and so $c_i = g\sigma + g^*\sigma^*$.

By Proposition 9 the unique p-chain b carried by $St(\eta,\tilde{X})$ such that $\partial b = \partial c_i$ is $b = -g[u,w,\partial\rho]$. Hence $\tilde{c}_i = g([w,\rho] - [u,\rho] - [u,w,\partial\rho])$ $= g\partial[u,w,\rho]$. Because $\pi \in$ supp $\tilde{c}_i$ if and only if π is a nonempty proper subset of $\{u,w\} * \rho = V(\text{supp } \tilde{c}_i)$ we have supp $\tilde{c}_i = \dot{\Delta}(V(\text{supp } \tilde{c}_i))$. Q.E.D.

6. The Proofs of Propositions 14 and 15.

Proposition 14. *If a graph G has a vertex covering subgraph family of generically* d*-rigid subgraphs that is vertex connected with multiplicity* d *then* G *is generically* d*-rigid.*

Proof. Suppose that G has a vertex covering subgraph family F of generically d-rigid subgraphs that is vertex connected with multiplicity d. Let $k = |F|$. Choose any subgraph in F and call it G_1. Set $A_1 = \{G_1\}$ and $B_1 = F\backslash A_1$. Note that $|A_1| = 1$ and that $\bigcup_{H\in A_1} H = G_1$ is generically d-rigid by hypothesis. We proceed recursively as follows:

Whenever $1 \le i \le k$ suppose that $\{A_i,B_i\}$ is a bipartition of F such that $|A_i| = i$ and $\bigcup_{H\in A_i} H$ is a generically d-rigid subgraph of G. Because F is vertex connected with multiplicity d there is a graph $G_{i+1} \in B_i$ with d vertices in common with some graph in A_i and so with $\bigcup_{H\in A_i} H$. By hypothesis G_{i+1} is generically d-rigid and so by Proposition 3 $(\bigcup_{H\in A_i} H) \cup G_{i+1}$ is generically d-rigid. Hence if we let $A_{i+1} = A_i \cup \{G_{i+1}\}$ and $B_{i+1} = F\backslash A_{i+1}$ we have that that $\{A_{i+1},B_{i+1}\}$ is a bipartition of F (unless $i+1 = k$) such that $|A_{i+1}| = i+1$ and $\bigcup_{H\in A_{i+1}} H$ is a generically d-rigid subgraph of G.

We conclude that $A_k = F$ and that $\bigcup_{H\in F} H$ is a generically d-rigid subgraph of G. Because F is vertex covering $\bigcup_{H\in F} H$ spans G. Thus by Proposition 2, G is generically d-rigid. Q.E.D.

<u>Proposition</u> <u>15</u>. *Let* X *be an abstract simplicial complex with* $\{u,w\} = \eta \in S^1(X)$. *For some* $p \geq 2$, *let* c *be a* p-*cycle on* X *with coefficients in some abelian group* Γ, *let* c' *be the restriction of* c *to* $\mathrm{Ast}(\eta,X)$, *and let* $\tilde{X}$ *be the* p-*completion of* X *over* η. *If* $\{c_1,c_2,c_3, \ . \ . \ . \ ,c_k\}$ *is a decomposition of* c' *modulo* $\mathrm{Susp}(\eta,\tilde{X})$, *then* $\sum_{i=1}^{k} \tilde{c}_i = c$, *where* $\tilde{c}_i$ *is the completion of* c_i *over* η *for* $1 \leq i \leq k$. *If, in addition,* c *is minimal and* G_i *denotes the* 1-*skeleton of* supp $\tilde{c}_i$ *for* $1 \leq i \leq k$, *then* $\{G_1,G_2,G_3, \ . \ . \ . \ ,G_k\}$ *is a vertex covering subgraph family of* $\bigcup_{i=1}^{k} G_i$ *that is vertex connected with multiplicity* p+1.

Proof. The cycle $\sum_{i=1}^{k} \tilde{c}_i$ equals the sum of the chain $\sum_{i=1}^{k} c_i = c'$ and some chain carried by $\mathrm{St}(\eta,\tilde{X})$. Because the completion of c' over η is unique, $\sum_{i=1}^{k} \tilde{c}_i = c$.

If, in addition, c is minimal, let $\{A,B\}$ be a bipartition of $\{1,2,3, \ . \ . \ . \ ,k\}$. Hence both A and B are nonempty. Thus we have that $0 \neq \sum_{i\in A} c_i \neq c'$ and $0 \neq \sum_{i\in B} c_i \neq c'$. It follows that $0 \neq \sum_{i\in A} \tilde{c}_i \neq c$ and $0 \neq \sum_{i\in B} \tilde{c}_i \neq c$. Because c is minimal $\sum_{i\in A} \tilde{c}_i$ cannot be a subcycle of c. Hence there is a p-simplex σ such that $0 \neq (\sum_{i\in A} \tilde{c}_i)(\sigma) \neq c(\sigma)$. Because $\sum_{i=1}^{k} \tilde{c}_i = c$ we have $0 \neq (\sum_{i\in B} \tilde{c}_i)(\sigma)$ also. Thus for some indices a in A and b in B we have $\tilde{c}_a(\sigma)$ and $\tilde{c}_b(\sigma)$ to be nontrivial. Because both supp $\tilde{c}_a$

and supp $\tilde{c}_b$ contain the p-simplex σ, $V(G_a)$ and $V(G_b)$ have at least p+1 common elements. Clearly then $\{G_1, G_2, G_3, \ldots, G_k\}$ is a vertex covering subgraph family of $\bigcup_{i=1}^{k} G_i$ that is vertex connected with multiplicity p+1.

Q.E.D.

7. The Proof of the Result.

<u>Theorem</u>. *The* 1-*skeleton of a minimal* (d-1)-*cycle complex,* $d \geq 3$, *is generically* d-*rigid.*

Proof. We use induction on the number of vertices in the complex. Recall we noted that if $d-1 > 0$, all (d-1)-cycle complexes have at least d+1 vertices; furthermore there are minimal (d-1)-cycle complexes with only d+1 vertices, and the 1-skeletons of these complexes are complete graphs, which are generically d-rigid by Proposition 1.

Suppose X is a minimal (d-1)-cycle complex with more than d+1 vertices and assume that the 1-skeleton of any minimal (d-1)-cycle complex with fewer vertices than X is generically d-rigid. Let c be a minimal (d-1)-cycle on X with coefficients in some abelian group Γ such that supp c = X. Let $\{u,w\} = \eta$ be any 1-simplex of X. Because $d-1 \geq 2$ we can form the (d-1)-completion of X over η, which we denote by $\tilde{X}$. Let c' be the restriction of c to $Ast(\eta,\tilde{X})$. It follows that $\partial c'$ is carried by $Susp(\eta,\tilde{X})$. Recall that c cannot be carried by $St(\eta, \text{supp } c) = St(\eta,X)$, so c' is nontrivial. Hence there is a maximal decomposition $D = \{c_1,c_2,c_3, \ldots ,c_k\}$ of c' modulo $Susp(\eta,\tilde{X})$.

Let $1 \leq i \leq k$. By the definition of D we know that c_i is a (d-1)-chain carried by $Ast(\eta,\tilde{X})$ and that ∂c_i is carried by $Susp(\eta,\tilde{X})$. By Proposition 10 the unique completion of c_i over η exists and we denote it by $\tilde{c}_i$. Let q be the labelling of $V(X) = V(\tilde{X})$ contracting η to w. Recall that c cannot be carried by $Ast(\eta, \text{supp } c) = Ast(\eta,X)$, so c' is a proper subchain of c, and c_i is a nontrivial proper subchain of c. Since c is a minimal cycle, c_i is not a cycle; recall that therefore η is a member of the support complex of the completion of c_i over η, i.e. $\eta \in \text{supp } \tilde{c}_i$. Thus q does contract an edge of supp $\tilde{c}_i$. Because D is maximal c_i is minimal modulo Susp$(\eta,\tilde{X})$ and hence by Proposition 13 $q_\#(\tilde{c}_i)$ is a minimal (d-1)-cycle on q(X) and either supp $q_\#(\tilde{c}_i)$ = q(supp $\tilde{c}_i$) or supp $\tilde{c}_i$ = $\dot{\Delta}$(V(supp $\tilde{c}_i$), the simplex boundary complex on its own vertex set.

Let G_i be the 1-skeleton of supp $\tilde{c}_i$. Then the 1-skeleton of q(supp $\tilde{c}_i$) is $q(G_i)$. If q(supp $\tilde{c}_i$) = supp $q_\#(\tilde{c}_i)$ then q(supp $\tilde{c}_i$) is a minimal (d-1)-cycle complex without the vertex u and so by the induction hypothesis $q_i(G_i)$ is generically d-rigid. Because supp $\tilde{c}_i$ is a (d-1)-cycle complex, η is an edge of at least 2 distinct (d-1)-simplices of supp $\tilde{c}_i$; thus η is an edge in at least d-1 distinct triangles of G_i. Thus by Theorem 4 the generic d-rigidity of $q(G_i)$ implies the generic d-rigidity of G_i. On the other hand, if supp $\tilde{c}_i$ = $\dot{\Delta}$(V(supp $\tilde{c}_i$)) then because d-1 > 0, G_i is a complete graph and so generically d-rigid.

By Proposition 15 $\sum_{i=1}^{k} \tilde{c}_i = c$; hence clearly $X = \text{supp } c \subset \bigcup_{i=1}^{k} \text{supp } \tilde{c}_i$. Thus $X^1 \subset \bigcup_{i=1}^{k} G_i \subset \tilde{X}^1$. But $X^1 = \tilde{X}^1$ so $X^1 = \bigcup_{i=1}^{k} G_i$. Furthermore, because c is minimal, by Proposition 15 $\{G_1, G_2, G_3, \ldots, G_k\}$ is a vertex covering subgraph family of $\bigcup_{i=1}^{k} G_i = X^1$ that is vertex connected with multiplicity d. Thus because G_i is generically d-rigid for $1 \le i \le k$, by Proposition 14, X^1 is generically d-rigid also.

The theorem follows by induction. Q.E.D.